Nuclear magnetic relaxation by the dipolar EMOR mechanism: Three-spin systems

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In aqueous systems with immobilized macromolecules, including biological tissue, the longitudinal spin relaxation of water protons is primarily induced by exchange-mediated orientational randomization (EMOR) of intra- and intermolecular magnetic dipole-dipole couplings. Starting from the stochastic Liouville equation, we have developed a non-perturbative theory that can describe relaxation by the dipolar EMOR mechanism over the full range of exchange rates, dipole couplings, and Larmor frequencies. Here, we implement the general dipolar EMOR theory for a macromolecule-bound three-spin system, where one, two, or all three spins exchange with the bulk solution phase. In contrast to the previously studied two-spin system with a single dipole coupling, there are now three dipole couplings, so relaxation is affected by distinct correlations as well as by self-correlations. Moreover, relaxation can now couple the magnetizations with three-spin modes and, in the presence of a static dipole coupling, with two-spin modes. As a result of this complexity, three secondary dispersion steps with different physical origins can appear in the longitudinal relaxation dispersion profile, in addition to the primary dispersion step at the Larmor frequency matching the exchange rate. Furthermore, and in contrast to the two-spin system, longitudinal relaxation can be significantly affected by chemical shifts and by the odd-valued (“imaginary”) part of the spectral density function. We anticipate that the detailed studies of two-spin and three-spin systems that have now been completed will provide the foundation for developing an approximate multi-spin dipolar EMOR theory sufficiently accurate and computationally efficient to allow quantitative molecular-level interpretation of frequency-dependent water-proton longitudinal relaxation data from biophysical model systems and soft biological tissue. © 2016 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0). [http://dx.doi.org/10.1063/1.4955423]

I. INTRODUCTION

For many years, the absence of a rigorous molecular theory of nuclear magnetic relaxation induced by magnetic dipole couplings in aqueous systems with immobilized macromolecules has prevented a reliable quantitative analysis of water-proton NMR (or MRI) relaxation data in terms of structure and dynamics of soft biological tissue or biophysical model systems, such as cross-linked protein gels. In such systems, protons exchange on a wide range of time scales between the bulk aqueous solution phase and the locally anisotropic macromolecular sites. This exchange affects proton relaxation by transferring magnetizations and coherences between the two environments. In addition, by randomizing the orientation of internuclear vectors, the exchange is also the motion that induces spin relaxation in the macromolecular sites. For this relaxation mechanism, known as exchange-mediated orientational randomization (EMOR), the motional-narrowing regime coincides with the fast-exchange regime. The conventional Bloch-Wangsness-Redfield (BWR) perturbation theory of nuclear spin relaxation\(^1\) therefore breaks down when, as is frequently the case, fast-exchange conditions do not prevail. Starting from the stochastic Liouville equation (SLE),\(^2,^3\) we have therefore developed a non-perturbative theory of relaxation induced by EMOR modulation of magnetic dipole-dipole couplings, valid without restrictions on exchange rate, dipole couplings, and Larmor frequencies.\(^4,^5\)

In the dipolar EMOR theory, we consider a system of \(m_A \geq 2\) mutually dipole-coupled spin-1/2 nuclei in the anisotropic (A) sites. This spin system comprises \(m_B\) labile spins, which exchange with the isotropic bulk (B) phase, and \(m_A - m_B\) nonlabile spins, which reside permanently in the A site. We distinguish two scenarios: symmetric exchange (\(m_B = m_A\)), where the spin system exchanges as an intact unit, and asymmetric exchange (\(m_B < m_A\)), where the spin system is fragmented by exchange. In asymmetric exchange, all multi-spin correlations that have developed as a result of dipole couplings between labile and nonlabile spins in the A site are lost,\(^5,^6\) leading to a qualitatively different relaxation behavior as compared to symmetric exchange. To identify different exchange cases, we use the notation “(spins in state A)–(spins in state B).” In previous reports in this series of papers, we have examined the symmetric case \(IS–IS\) (Paper I\(^4\)) and the asymmetric case \(IS–I\) (Paper II\(^5\)).

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Here, we implement the general dipolar EMOR theory, presented in Paper II, for a three-spin system, examining the symmetric exchange case $ISP–ISP$ and the asymmetric cases $ISP–IS$ and $ISP–I$. The exchanging entity in these cases might be a hydronium ion binding to a specific site in a protein ($ISP–ISP$), a water molecule temporarily occupying an internal cavity in the protein ($ISP–IS$), or a labile hydroxyl proton in an amino acid side-chain ($ISP–I$). As compared to the previously analyzed two-spin system, several new features emerge for the three-spin system. Because there are now three dipole couplings rather than one, relaxation is affected by distinct correlations (as well as self-correlations) and the magnetizations are dynamically coupled to two-spin and three-spin modes. Furthermore, in the $ISP–I$ case, one of the three dipole couplings is static. As a result of these complications, three secondary dispersion steps (some of them inverted) with different physical origins can appear, under different conditions, in the longitudinal relaxation dispersion profile, in addition to the primary dispersion step where the Larmor frequency matches the exchange rate. Furthermore, and in contrast to the two-spin system, the longitudinal relaxation can be significantly affected by chemical shifts and by the odd-valued (“imaginary”) part of the spectral density function.

The outline of this paper is as follows. In Sec. II, we develop the general dipolar EMOR theory for a three-spin system. Once the matrix elements of the Zeeman and dipolar Liouvilleans in the 63-dimensional three-spin Liouville space have been calculated, it is straightforward to implement the general dipolar EMOR theory of Paper II. In Sec. III, we develop a limiting form of the dipolar EMOR theory, based on the BWR master equation, and valid in the motional-narrowing regime. The semi-analytical results obtained here are used to rationalize the rich variety of relaxation behavior exhibited by the three-spin system. In Sec. IV, we illustrate the theory developed in Secs. II and III by numerical results, emphasizing the new features that emerge at the three-spin level. Lengthy derivations and tables are relegated to eight appendices.

II. GENERAL RESULTS

A. System, model, and solution

We consider a system of three mutually dipole-coupled spin-1/2 nuclei, denoted $I$, $S$, and $P$, some or all of which exchange between a solid-like anisotropic (A) state and a liquid-like bulk (B) state. The spins may be homonuclear or heteronuclear and the chemical shifts may differ between states A and B. State A comprises an isotropic distribution of sites, labelled $\alpha = 1, 2, \ldots$, each of which has a fixed orientation. At any time, a fraction $P_A$ of the labile spins reside in state A, while a fraction $P_B = 1 - P_A$ reside in state B. In most applications of interest, $P_A \ll 1$, a condition that we refer to as the dilute regime. We consider three exchange cases.

In the symmetric case, denoted $ISP–ISP$, the three spins exchange as an intact unit, for example, the three protons in a hydronium ion, $\text{H}_3\text{O}^+$, or in an acetate ion, $\text{CH}_3\text{COO}^-$. In the asymmetric $ISP–I$ case, the labile spin $I$ exchanges with state B, whereas the nonlabile spins $S$ and $P$ remain in state A so their mutual dipole coupling is static. An example of this case is a serine side-chain, with spin $I$ identified as the labile hydroxyl proton and spins $S$ and $P$ identified as the adjacent nonlabile methylene protons.

In the asymmetric $ISP–IS$ case, the labile spins $I$ and $S$ exchange as a unit, leaving the nonlabile spin $P$ (without any dipole couplings) in state A. An example of this case is an internal water molecule, where, in the A sites, the two water protons $I$ and $S$ are also dipole-coupled to a nonlabile protein proton $P$.

In the EMOR model, the orientations of all internuclear vectors involving at least one labile spin are instantaneously randomized upon exchange, thereby inducing dipolar relaxation. This assumption is justified if the mean survival time of the labile spin(s) in the A sites is long compared to the time required for orientational randomization when the labile spin(s) has been transferred to state B. We can then ignore all dipole couplings among the labile spins in state B. If so desired, the small and frequency-independent relaxation contribution from fast modulation of dipole couplings in state B can be added to the final expression for the overall relaxation rate, as described in Paper I.

The Zeeman Hamiltonian in state A is

$$H_Z^A = \omega_I^A I_z + \omega_S^A S_z + \omega_P^A P_z,$$

and similarly for state B. The “chemical shifts” are defined with reference to spin $I$ so $\delta_I^A \equiv 0$ and $\delta_i^A \equiv (\omega_i^A - \omega_I^A)/\omega_I^A$ for $S = S$ or $P$. As noted in Paper I, scalar couplings among non-isochronous spins in state A are neglected here because they are invariably much smaller than the corresponding dipole couplings. For the $ISP–ISP$ and $ISP–IS$ cases, scalar couplings among non-isochronous spins in state B would affect the relaxation behavior, but, for the applications that we have in mind (see above), molecular symmetry ensures that the proton spins are isochronous in state B.

The dipolar Hamiltonian for A-site $\alpha$ is

$$H_D^\alpha = H_D^{\alpha, IS} + H_D^{\alpha, IP} + H_D^{\alpha, SI},$$

with (X) denotes either of the three spin pairs $IS$, $IP$, or $SP$)

$$H_D^{\alpha, X} = -\frac{2}{\sqrt{3}} \omega_{\alpha, X} \sum_{M = -2}^{2} T_{1M}(X) V_{M,0}^\alpha(\Omega_X^n).$$

Here, $T_{1M}(X)$ are orthonormal three-spin irreducible spherical tensor operators (see below), $V_{M,0}^\alpha(\Omega_X^n)$ are rank-2 Wigner functions, and $\Omega_X^n$ are the Euler angles that specify the orientation of the internuclear vector $r_X$ in site $\alpha$ with respect to the lab-fixed frame.

Rather than using the three sets of Euler angles $\Omega_X^n$, it is more convenient to specify the internuclear geometry by the three interior angles $\beta_1$, $\beta_2$, and $\beta_3$ of the ISP triangle (Fig. 1) and then specify the orientation of this triangle with respect to the lab-fixed frame by the Euler angles $\Omega^n \equiv (\psi^n, \theta^n, \phi^n)$. We adopt the following convention: $\psi^n$ is the angle of rotation of the vector $r_{IS}^n$ about the lab-fixed $z_1$ axis (parallel to the $B_1$ field), $\theta^n$ is the angle between the positive $z_L$ axis and $r_{IS}^n$, and $\phi^n$ is the angle of rotation of the triangle plane about the
The internuclear geometry, which is the same in all sites $\alpha$, is fully determined by specifying two of the three interior angles (which sum up to 180°) and one of the three dipole coupling frequencies $\omega_{D,1S}$, $\omega_{D,IP}$, and $\omega_{D,SP}$ (which depend on the internuclear separations $r_{xy}$). If we specify the

$$\omega_{D,1S} \equiv \frac{3}{2} \left( \frac{\mu_0}{4\pi} \right) \frac{\gamma_I \gamma_S \hbar}{r_{1S}^3},$$

(5)

then the remaining two dipole couplings are given by

$$\omega_{D,IP} = \left[ \frac{\sin(\beta_I + \beta_S)}{\sin \beta_S} \right]^3 \omega_{D,1S},$$

(6a)

$$\omega_{D,SP} = \left[ \frac{\sin(\beta_I + \beta_S)}{\sin \beta_I} \right]^3 \omega_{D,1S}.$$  

(6b)

In the general case of a non-degenerate triangle ($\beta_I \neq \beta_S \neq \beta_P$), the three dipole couplings are all different. In the special case of an isosceles triangle with $\beta_I = \beta_S$ (so spin $P$ is at the apex), Eq. (6) reduces to $\omega_{D,IP} = \omega_{D,SP} = 8 \sin^3(\beta_P/2)\omega_{D,1S}$. For an equilateral triangle ($\beta_I = \beta_S = \beta_P = 60^\circ$), of course, all three dipole couplings are equal.

The composite spin density operator $\sigma(t)$ evolves according to the stochastic Liouville equation (SLE)

$$\frac{d}{dt} \sigma(t) = (\mathcal{W} - i \mathcal{L}) \sigma(t),$$

(7)

where the Liouvillian $\mathcal{L} \equiv [H, \ldots]$ is the derivation superoperator corresponding to the Hamiltonian in each site: $H^0_B$ in state $B$ and $H^0_S + H^0_D$ in $S$. The exchange superoperator $\mathcal{W}$ describes the transfer of labile spins from state $B$ to $S$ or vice versa (direct exchange between $A$-sites is not allowed in the EMOR model), and the consequent instantaneous switching of the spin Hamiltonian. For the asymmetric cases $ISP-I$ and $ISP-IS$, the spin system is fragmented by the exchange. Consequently, all multi-spin correlations that have developed as a result of dipole couplings between labile and nonlabile spins in state $A$ are lost. To describe both of these effects, we decompose the exchange superoperator as

$$\mathcal{W} = \mathcal{T}_m \otimes \mathcal{T}_s - \mathcal{K}_m \otimes \mathcal{K}_s.$$  

(8)

The “molecular” operators $\mathcal{T}_m$ and $\mathcal{K}_m$ act on the site kets $|\alpha\rangle$, so their composite-space supermatrix representations are block-diagonal with respect to the spin operators. These operators define the kinetic model (site-to-site transition probabilities), regardless of whether the spin system is fragmented or not. The superoperators $\mathcal{T}_s$ and $\mathcal{K}_s$ act on spin operators, so (as for $\mathcal{L}$) their composite-space supermatrix representations are block-diagonal in the site basis. These superoperators distinguish labile from nonlabile spins and they account for decorrelation of multi-spin modes by exchange fragmentation of the spin system.

Macroscopic spin observables are related to a density operator summed over all sites,

$$\langle \sigma(t) \rangle \equiv \sum_{\alpha=0}^{N} \sigma^\alpha(t) = \sigma^B(t) + \sigma^A(t),$$

(9)

with the initial value

$$\langle \sigma(0) \rangle = P_B \eta^B + P_A \eta^A.$$  

(10)

The operators $\eta^A$ and $\eta^B$, which act in spin Liouville space, depend on the initial condition of the spin system (selective or nonselective excitation) and, for heteronuclear spin systems, on the relative magnetogyric ratios.

The Laplace-transformed density operator $\hat{\sigma}(s)$, represented in spin Liouville space as a partitioned column vector, can be computed from the (exact) supermatrix equation

$$\hat{\sigma}^B(s) = \hat{U}^{BB}(s) \hat{U}^{BA}(s) \hat{U}^{AA}(s) \eta^B,$$

(11)

with the Laplace-transformed site-averaged resolvent matrices given by

$$\hat{U}^{BB}(s) = \tau_B P_B [G^B(s)^{-1} - T G^A(s) T']^{-1},$$

(12a)

$$\hat{U}^{AA}(s) = \tau_A P_A [G^A(s)^{-1} - T G^B(s) T']^{-1},$$

(12b)

$$\hat{U}^{BA}(s) = \frac{P_A}{P_B} \hat{U}^{BB}(s) T G^A(s),$$

(12c)

$$\hat{U}^{AB}(s) = \frac{P_B}{P_A} \hat{U}^{AA}(s) T' G^B(s).$$

(12d)

Here, $\tau_A$ and $\tau_B$ are the mean survival times in the two states, related by the detailed balance condition $P_A \tau_B = P_B \tau_A$. The dimensions of the spin Liouville subspaces, excluding the physically irrelevant identity operator, are for state $A$ $D_A = 63$ and for state $B$ $D_B = 5$ ($ISP-I$ case), $D_B = 15$ ($ISP-IS$), or $D_B = 63$ ($ISP-ISP$). In Eq. (12), we have also introduced the supermatrices

$$G^B(s) \equiv [(1 + s\tau_B) \mathbf{1} + i L^B Z \tau_B]^{-1}$$

(13)

and

$$G^A(s) \equiv \{ s \tau_A \mathbf{1} + K + i L^A T \tau_A + i L D(\Omega) \tau_A \}^{-1},$$

(14)
where $\mathbf{L}_Z$ and $\mathbf{L}_D(\Omega)$ are the Liouvillian supermatrices corresponding to the Zeeman and dipolar Hamiltonians, respectively, and the angular brackets indicate an isotropic orientational average.

To make full use of symmetry, we represent spin Liouville space in a basis of irreducible spherical tensor operators (ISTOs) $B_{\lambda} \equiv T^Q_{\lambda}(l)$ of rank $K$, quantum order $Q$, and additional quantum numbers $\lambda$. The explicit form of the ISTOs in the three bases (of dimension 3, 15 and 63) used here is given in Appendix A of the supplementary material.

All of these operators are normalized in three-spin Liouville space. Explicit expressions for the elements of the Zeeman and dipolar Liouvillian supermatrices $\mathbf{L}_Z$ and $\mathbf{L}_D$ are derived in Appendices B and C, respectively.

In Eqs. (12) and (14), $\mathbf{T}$ and $\mathbf{K}$ are the supermatrix representations of the spin superoperators $T_{\lambda}$ and $K_{\lambda}$ in Eq. (8). For the symmetric $ISP-ISP$ case, all spins are labile so there is no exchange fragmentation. Consequently, $\mathbf{T} = \mathbf{K} = \mathbf{I}$, the $63 \times 63$ identity matrix.

\[ T_{nBpA} = \delta_{nB1} \delta_{pA1} + \delta_{nB2} \delta_{pA2} + \delta_{nB3} \delta_{pA11} + \delta_{nB4} \delta_{pA17} + \delta_{nB5} \delta_{pA14} + \delta_{nB6} \delta_{pA20} + \delta_{nB7} \delta_{pA21} + \delta_{nB8} \delta_{pA29} + \delta_{nB9} \delta_{pA32} + \delta_{nB10} \delta_{pA35} + \delta_{nB11} \delta_{pA36} + \delta_{nB12} \delta_{pA44} + \delta_{nB13} \delta_{pA47} + \delta_{nB14} \delta_{pA53} + \delta_{nB15} \delta_{pA59}. \]

The three one-spin modes in state A that do not involve either of the labile spins $I$ and $S$ are $n_A = 3, 22,$ and $37$ (Table S1), so

\[ K_{nAPA} = \delta_{nAPA}[1 - \delta_{nA2} - \delta_{nA22} - \delta_{nA37}]. \]

The angular brackets in Eq. (14) signify an average over the isotropic distribution of site orientations, $f(\Omega) = 1/(8\pi^2)$. Because $\mathbf{G}^A$ is isotropically averaged, it must reflect the axial symmetry in spin Liouville space, imposed by the external magnetic field. For a basis of ISTOs $T^Q_{\lambda}(l)$, it then follows from the Wigner-Eckart theorem that $\mathbf{G}^A$ and the resolvent matrices in Eq. (12) are block-diagonal in the projection index $Q$. Longitudinal relaxation can therefore be fully described within the 19-dimensional zero-quantum subspace. However, for the asymmetric $ISP-I$ case, exchange interconverts the three one-spin modes in state B, $n_B = 1, 2,$ and $3$ (Table S3) and the corresponding three one-spin modes in state A, $p_A = 1, 20,$ and $35$ (Table S1). Consequently,\n
\[ T_{nBpA} = \delta_{nB1} \delta_{pA1} + \delta_{nB2} \delta_{pA20} + \delta_{nB3} \delta_{pA35}. \]

Identifying the 15 one- and two-spin modes in state A that do not involve the labile I spin (Table S1), we obtain

\[ K_{nAPA} = \delta_{nAPA}[1 - \delta_{nA2} - \delta_{nA22} - \delta_{nA37}]. \]

For the asymmetric $ISP-IS$ case, exchange interconverts the 15 modes in state B (Table S2) and the corresponding one- and two-spin modes in state A (Table S1).

\[ \mathbf{R}_1 = \frac{P_B \sum_{nB} \eta^B_{nB} + P_A \sum_{nA} \eta^A_{nA}}{\sum_{nB} \sum_{nB'} U^B_{nBP} + \sum_{nA} \sum_{nA'} U^A_{nP A'}}, \]

where we have made use of Eqs. (10) and (11) and introduced the shorthand notation $U^X_{np} = \alpha[|X\rangle\langle X|](\alpha/p)$. The primed sums in Eq. (19) run over spin modes (or basis operators) corresponding to the longitudinal magnetization of the observed spin(s). In the following, we specify the observed spin(s) by a subscript, e.g., $\mathbf{R}_{1IS}$.

The elements of the vectors $\eta^B$ and $\eta^A$ depend on the initial conditions for the relaxation experiment. For nonselective excitation, which always applies to field-cycling experiments,\n
\[ \eta^A_{nA} = \delta_{nA1} + \kappa_S \delta_{nA2} + \kappa_P \delta_{nA3}, \quad \text{(ISP-I)} \]

\[ \eta^B_{nB} = \begin{cases} \delta_{nB1} + \kappa_S \delta_{nB2} \quad \text{(ISP-IS)}, \\ \delta_{nB1} + \kappa_S \delta_{nB2} + \kappa_P \delta_{nB3} \quad \text{(ISP-ISP)}. \end{cases} \]

Here, the relative magnetogyric ratios $\kappa_S \equiv \gamma_S/\gamma_I$ and $\kappa_P \equiv \gamma_P/\gamma_I$ account for the relative magnitude and sign
of the equilibrium spin density operator components in the high-temperature approximation. For selective excitation of the labile spin(s), \( \eta_s^A = \eta_s^B \) with \( \eta_s^B \) as in Eq. (20b). These two initial conditions will be indicated by a superscript, e.g., \( \tilde{R}_{11,1}^{sl} \) and \( \tilde{R}_{11,1}^{SL} \). Explicit expressions for the ILRR, based on Eq. (19), are given in Appendix D for the various combinations of excited, observed, and labile spin(s).

In the dilute regime, where \( P_A \ll 1 \), all matrix elements in the denominator of Eq. (19) except \( U_{npnB}^{BB} \) can be neglected if the observed magnetization includes at least one labile spin. Furthermore, in the numerator, the second term (proportional to \( P_A \)) can then be neglected. The ILRR is then the same for nonselective excitation as for selective excitation of the labile spin(s). We therefore use the superscript “dil” without specifying the excitation mode. For the ISP–I case, we obtain

\[
\tilde{R}_{1,1}^{dil} = \tilde{R}_{1,1}^{dil} = \tilde{R}_{1,1}^{dil} = [U_{11}^{BB} + \kappa_s U_{12}^{BB} + \kappa P U_{13}^{BB}]^{-1},
\]

\[
\tilde{R}_{1,1,IS}^{dil} = (1 + \kappa_S)[U_{11}^{BB} + U_{21}^{BB} + \kappa S (U_{12}^{BB} + U_{22}^{BB}) + \kappa P (U_{13}^{BB} + U_{23}^{BB})]^{-1},
\]

\[
\tilde{R}_{1,1,ISP}^{dil} = (1 + \kappa_S + \kappa P)[U_{11}^{BB} + U_{21}^{BB} + U_{31}^{BB} + \kappa S (U_{12}^{BB} + U_{22}^{BB} + U_{32}^{BB}) + \kappa P (U_{13}^{BB} + U_{23}^{BB} + U_{33}^{BB})]^{-1}.
\]

According to Eq. (12a), \( U_{npnB}^{BB} \propto 1/P_A \) in the dilute regime. It then follows from Eqs. (21)–(23) that the ILRR is strictly proportional to \( P_A \). For the ISP–I case, the \( 3 \times 3 \) matrix \( U^{BB}(0) \) is diagonal and, as shown in Appendix D, Eq. (21) yields

\[
\tilde{R}_{1,1}^{dil} = \tilde{R}_{1,1}^{dil} = \tilde{R}_{1,1}^{dil} = \frac{P_A}{\tau_A}(1 - g_{11}),
\]

with the shorthand notation \( g_{np} = \langle n \rangle [G(n)] \langle p \rangle \). This result is of the same form as for the asymmetric two-spin case IS–I considered in Paper II.

C. Nuclear permutation symmetry

In the EMOR model, all labile spins are affected in the same way by the relaxation-inducing dynamics, since all dipole couplings involving labile spins are completely randomized in orientation by the exchange. The symmetry of the ILRR under nuclear permutation is therefore determined solely by the internuclear geometry, that is, the lengths of the internuclear vectors and the angles between them (Fig. 1). Consequently, the ILRR is invariant under any permutation of the observed labile spins. This is generally true for the dipolar EMOR model, even for non-dilute conditions and for multi-spin systems. For the three-spin system considered here, \( \tilde{R}_{1,1,SP}^{ISP–ISP} \) is invariant under any permutation of the three spins \( I, S, \) and \( P \), whereas \( \tilde{R}_{1,1,ISP}^{ISP–ISP} \) and \( \tilde{R}_{1,1,ISP}^{ISP–IS} \) are invariant under \( I \leftrightarrow \Delta \) interchange.

III. MOTIONAL-NARROWING REGIME

The results of Sec. II allow us to calculate the ILRR \( \tilde{R}_I \) for any combination of excited, observed, and labile spin(s). With modest effort, we can compute the complete relaxation dispersion profile \( R_1(\omega_B) \) for arbitrary values of the mean survival time \( \tau_A \), dipole coupling constant \( \omega_{D,ISP} \), internuclear geometry \( \beta_I \) and \( \beta_S \), and chemical shifts in the A and B states. The main virtues of the SLE-based approach are its generality and computational efficiency, but it does not necessarily provide an understanding of the physical basis of the computed relaxation behavior. In this section, we shall therefore obtain the ILRR by a different approach, based on the semi-classical BWR perturbation theory. Although restricted to the motional-narrowing (MN) regime, where \( \omega_D \tau_A \ll 1 \), the BWR approach is more physically transparent. (As discussed in Appendix E, we assume that all three dipole couplings satisfy this inequality, including the static SP coupling in the ISP–I case.) In addition, the requirement that the SLE- and BWR-based results coincide in the MN regime provides a valuable check.

In the MN regime, the composite spin density operator evolves according to the “stochastic Redfield equation” (SRE)

\[
\frac{d}{dt} \sigma(t) = [\mathcal{W} - i \mathcal{L}_Z - i \Delta - \mathcal{R}] \sigma(t),
\]

where \( \mathcal{W} \) is the same exchange superoperator (8) as in the SLE (7), \( \mathcal{L}_Z \) is the Zeeman Liouvillian, and \( \mathcal{R} \) is the relaxation superoperator prescribed by BWR theory. The superoperator \( \Delta \equiv \mathcal{L}_D,SP \) is associated with the time-independent dipolar coupling between spins \( S \) and \( P \) in the ISP–I case. For the other two exchange cases, this term is absent. For the EMOR model, the SRE (25) can be solved in site space in the same manner as the SLE (7). Specifically, Eqs. (11) and (12) remain valid, but Eq. (14) is replaced by
\[ G^{\alpha}(s) = \left\{ s \tau_A I^A + K + i L^A_z \tau_A + i \Lambda^\alpha \tau_A + R^\alpha \tau_A \right\}^{-1}, \]

(26)

where \( R^\alpha \) is the orientation-dependent relaxation supermatrix for site \( \alpha \) (Appendix E).

We use the same notation as in Paper II for relaxation supermatrix elements between basis operators that involve a single spin. For example, \( R^\alpha \tau_A = (\{ \eta [R^\alpha] \} P_A) = -2^{3/2} (\{ \eta [R^\alpha] \} P_A), \) with the basis operators numbered as in Table S1.\(^8\) These local relaxation rates are of four kinds. First, there are longitudinal (\( \eta [K] \) and \( \eta [S] \)) and transverse (\( R^\alpha \tau_A \) and \( R^\alpha \tau_B \)) auto-spin auto-mode rates. Second, there are longitudinal (\( R^\alpha \tau_A \) and \( R^\alpha \tau_B \)) and transverse (\( R^\alpha \tau_A \) and \( R^\alpha \tau_B \)) auto-spin cross-mode rates. Third, there are the auto-spin cross-mode rates \( R^\alpha \tau_A \) and \( R^\alpha \tau_B \) and the corresponding rates for spin \( S \) and spin \( P \). Finally, there are cross-spin cross-mode rates, like \( R^\alpha \tau_A \). All these rates pertain to a particular site \( \alpha \) and they therefore depend on the orientation of the dipole vectors in that site as detailed in Appendix E.\(^8\)

In addition to these single-spin-mode rates, \( R^\alpha \) contains local relaxation rates connecting modes involving two or three spins. The two-spin modes have even parity under spin inversion conjugation (SIC)\(^7,13,14\) whereas the single-spin and three-spin modes have odd parity. The relaxation supermatrix \( R^\alpha \) in the ISTO basis is therefore block-diagonal; that is, it does not couple modes of different parity (Appendix E).\(^8\) The longitudinal relaxation behavior (and the ILRR) is therefore affected by local relaxation rates involving two-spin modes only in the ISP–I case, where \( \Delta^\alpha \) couples the odd and even blocks of \( R^\alpha \) (Appendix E).\(^8\) The three-spin modes are destroyed by fragmentation of the spin system in the asymmetric exchange cases so the ILRR is affected by relaxation rates involving three-spin modes only in the symmetric ISP–ISP case.

To sum up, the local relaxation rates that affect the ILRR are of types (1-spin|\( \eta [R^\alpha]|1\)-spin) and (2-spin|\( \eta [R^\alpha]|2\)-spin) for the ISP–I case; of type (1-spin|\( R^\alpha|1\)-spin) only for the ISP–IS case; and of types (1-spin|\( R^\alpha|1\)-spin), (1-spin|\( R^\alpha|3\)-spin), and (3-spin|\( R^\alpha|3\)-spin) for the ISP–ISP case.

We shall carry out the BWR treatment for three homonuclear spins in the dilute regime, which are also the conditions of primary interest for applications. Consequently, we have \( \kappa_S = \kappa_P = 1 \) and \( P_A \ll 1 \). The ILRR in the dilute regime, given by Eqs. (21)–(24) for the different exchange cases, only involves elements of the supermatrix \( \hat{U}^{BB}(0) \).

Combining Eqs. (12a) and (13) and the detailed balance relation \( P_{A1} \tau_B = P_{B1} \tau_A \approx \tau_A \), we obtain

\[ \hat{U}^{BB}(0) = \frac{\tau_A}{P_A} [I^B + i L^B_z \tau_B - T G^A(0) T']^{-1}, \]

(27)

with \( G^A(0) \) given by Eq. (26). Because the supermatrix \( G^A(0) \) is isotropically averaged, it must reflect the axial symmetry in spin Liouville space. According to the Wigner-Eckart theorem,\(^12\) \( G^A(0) \) must then be block-diagonal in the projection index \( Q \) of the ISTO basis of state \( A \) (Table S1).\(^8\)

The supermatrix \( T \) connects spin modes in states A and B that are transferred by exchange. It is clear, therefore, that the supermatrix \( T G^A(0) T' \) is block-diagonal in the projection index \( Q \) of the ISTO basis of state B (Appendix A),\(^8\) as is \( I^B \). Because the ILRR is determined by elements from the \( Q = 0 \) block of \( \hat{U}^{BB}(0) \) and since the block-diagonal structure is maintained under inversion, we need only retain the \( Q = 0 \) block of the matrices in Eq. (27). The dimension of this block is 1, 5, and 19 for exchange case ISP–I, ISP–IS, and ISP–ISP, respectively (Appendix A). Further simplification can be achieved by considering the SIC parity\(^7,13,14\) of the supermatrix \( G^A(0) \). We now consider the ILRR in the MN regime for the three exchange cases, in order of increasing complexity. For exchange cases ISP–IS and ISP–I, we only sketch the derivation, which is fully reproduced in Appendix F.\(^8\)

In Subsections III A–III D, we restrict the treatment to isochronous spins. The Zeeman Liouvillian in Eq. (27) can then be dropped, because the \( Q = 0 \) block of \( I^B \) is a null matrix. Explicit expressions for the elements of the supermatrices \( \Delta^\alpha \) and \( R^\alpha \) are derived in Appendix E.\(^8\) In Sec. IV D and Appendix G,\(^8\) we generalize the BWR treatment to include the effects of chemical shifts.

### A. Exchange case ISP–ISP

In this case, \( T \) and \( K \) are identity matrices and there is no static dipole coupling, so Eqs. (26) and (27) yield

\[ \hat{U}^{BB}(0) = \frac{\tau_A}{P_A} [1 - G^A(0)]^{-1} \]

(28)

and

\[ G^A(0) = \left( \{ I + R^\alpha \tau_A + i L^A_z \tau_A \} \right)^{-1} \]

\[ = \left( 1 + i L^A_z \tau_A \right)^{-1} - \left( 1 + i L^A_z \tau_A \right)^{-1}, \]

(29)

where, consistent with the MN approximation, we have expanded to second order in \( \omega \tau_A \) (that is, to first order in \( |R^\alpha| \tau_A \)). In Eqs. (28) and (29), all matrices refer to the \( 19 \times 19 \) block of the full \( 63 \times 63 \) supermatrices. Because the \( Q = 0 \) block of \( I^B \) is a null matrix, we obtain

\[ \hat{U}^{BB}(0) = \frac{1}{P_A} \left( R^\alpha \right)^{-1}. \]

(30)

The desired ILRR can now be obtained by inserting the required supermatrix elements from Eq. (30) into Eq. (23). For example,

\[ \tilde{R}_{\alpha}^{BB} = P_A \left( \frac{1}{3} \sum_{n_1 = 1}^{3} \sum_{p=1}^{3} \left( R^\alpha \right)^{-1}_{np} \right)^{-1}. \]

(31)

For clarity, the summand in Eq. (31) is the \((np)\)-element of the inverse of the \( 19 \times 19 \) block of the isotropically averaged relaxation supermatrix \( (R^\alpha) \), given explicitly by Eq. (E.17).\(^8\) Actually, since the relaxation supermatrix \( R^\alpha \) itself is block-diagonal with respect to the SIC parity of the basis operators (Appendix E),\(^8\) we need only invert the \( 10 \times 10 \) odd-parity \( Q = 0 \) block of \( (R^\alpha) \) corresponding to the subspace spanned by the first ten basis operators in Table S1 A.\(^8\) These ten spin modes comprise the three single-spin longitudinal
(L) magnetizations and seven three-spin zero-quantum coherences (ZQCs), so the $10 \times 10$ matrix can be further partitioned as

$$
\langle R^{\alpha} \rangle = \begin{bmatrix}
R_L & R_{L/ZQC} & R_{ZQC/L} & R_ZQC
\end{bmatrix}.
$$

(32)

The $3 \times 3$ longitudinal block of the inverse $\langle R^{\alpha} \rangle^{-1}$ in Eq. (31) can then be expressed as

$$
\langle R^{\alpha} \rangle^{-1} = \begin{bmatrix}
R_L - R_{L/ZQC} & R_{ZQC/L} & R_ZQC
\end{bmatrix}^{-1}.
$$

(33)

The elements of $R_L$ are auto-mode rates produced exclusively by self-correlations. In contrast, the elements of $R_{L/ZQC}$ and $R_{ZQC/L}$ are cross-mode rates produced exclusively by distinct correlations, corresponding to $X \neq Y$ in Eq. (E.17). Because all relaxation rates are positive, it follows from Eqs. (31) and (33) that the distinct correlations make a negative contribution that reduces the ILRR: $\tilde{R}_{1,ISP} \leq \tilde{R}_{1,ISP}$. Relaxation coupling to the three-spin ZQCs occurs only in the $ISP-ISP$ case, where the three spins exchange as an intact unit.

B. Exchange case $ISP-ISP$

In this case, $\tilde{U}^{BB}(0)$ in Eq. (27) is a $5 \times 5$ matrix in the subspace spanned by the first five basis operators (with $Q = 0$) in Table S2.8 Since there is no static dipole coupling, the block-diagonality of $R^z$ with respect to SIC parity carries over to $G^y(0)$ and $U^{BB}(0)$. To obtain the ILRR, we therefore need to consider only the $2 \times 2$ odd-parity $Q = 0$ subspace spanned by the first two basis operators $I_z/\sqrt{2}$ and $S_z/\sqrt{2}$, so that

$$
\tilde{U}^{BB}(0) = \begin{bmatrix}
\frac{\tau_A}{P_A} & 1 - g_{11} - g_{12} \\
1 - g_{21} & 1 - g_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\tau_A}{P_A} & 1 \\
1 - g_{11} - g_{22} + g_{11}g_{22} - g_{12}g_{21} & 1 - g_{22} + g_{12}g_{21} + g_{11}g_{22}
\end{bmatrix}
$$

(34)

where, as before, $g_{np} \equiv \langle n|G^y(0)|p \rangle$. The desired ILRR can now be obtained by inserting the required supermatrix elements from Eq. (34) into Eq. (25). For example,

$$
\tilde{R}_{1,ISP} = \frac{P_A}{\tau_A} \frac{2(1 - g_{11} - g_{22} + g_{11}g_{22} - g_{12}g_{21})}{(2 - g_{11} - g_{22} + g_{12} + g_{21})}.
$$

(35)

Evaluating the four matrix elements $g_{np}$ in Eq. (35), we obtain (Appendix F)

$$
\tilde{R}_{1,ISP} = \frac{P_A}{\tau_A} \frac{2(\Gamma_{11} + \Gamma_{13} - \Gamma_{12} + \Gamma_{14})}{(\Gamma_{11} + \Gamma_{13} - \Gamma_{12} - \Gamma_{14})}.
$$

(36)

where

$$
\begin{bmatrix}
\Gamma_{11} & \Gamma_{13} & \Gamma_{12} & \Gamma_{14}
\end{bmatrix} = \begin{bmatrix}
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\end{bmatrix}.
$$

(37)

The elements of the last matrix are obtained by orientational averaging of the “cross relaxation” rates

$$
\begin{bmatrix}
\Gamma_{11} & \Gamma_{13} & \Gamma_{12} & \Gamma_{14}
\end{bmatrix} = \begin{bmatrix}
R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} \\
R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} \\
R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} \\
R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} & R^{TF}_{zz} \\
\end{bmatrix}^{-1}
\times \begin{bmatrix}
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle & \langle R^{TF}_{zz} \rangle \\
\end{bmatrix}.
$$

(38)

where $\omega_0$ is the common Larmor frequency of the three isochronous spins in state A. The local relaxation rates that appear in Eq. (38) are given in explicit form in Eqs. (E.21)–(E.25). They include cross-spin rates, such as $R^{TF}_{zz}$, as well as cross-mode rates, such as $R^{TF}_{zz}$. But because all these rates connect single-spin modes, they only involve dipolar self-correlations, as follows from the selection rule (E.15).

As shown in Appendix F,8 the $ISP-ISP$ result in Eq. (36) reduces to the previously obtained4,5 ILRR for the symmetric two-spin case $IS-ISP$ if spin P is removed and to the previously obtained6 ILRR for the asymmetric two-spin case $IP-IP$ if spin P is removed.

In the secular approximation, valid for $\omega_0 \gg \omega^2 \tau_A$, all cross-mode rates vanish and Eqs. (36)–(38) yield (Appendix F)

$$
\tilde{R}_{1,ISP} = \frac{P_A}{\tau_A} \frac{2r_{11}r_{13} - r_{14}^2}{(r_{11} + r_{13} - 2r_{14})}.
$$

(39)

where

$$
\begin{align}
r_{11} &= \langle R^{TF}_{zz} \rangle = \frac{\langle R^{TF}_{zz} \rangle^2}{R^{TF}_{zz}}, \\
r_{13} &= \langle R^{TF}_{zz} \rangle = \frac{\langle R^{TF}_{zz} \rangle^2}{R^{TF}_{zz}}, \\
r_{14} &= \langle R^{TF}_{zz} \rangle = \frac{\langle R^{TF}_{zz} \rangle^2}{R^{TF}_{zz}}.
\end{align}
$$

(40, 41, 42)

In the secular approximation, the ILRR is thus seen to be fully determined by the six unique longitudinal auto-mode rates.

C. Exchange case $ISP-IP$

The dilute-regime ILRR for the $ISP-IP$ case, given by Eq. (24), only involves the element $g_{11}$ of the supermatrix $G^y(0)$ defined by Eq. (26), where the supermatrix $\Omega^*$ associated with the static SP dipole coupling now must be reckoned with. Evaluating the element $g_{11}$, we find (Appendix F)

$$
\tilde{R}_{1,ISP} = \frac{P_A}{\tau_A} \frac{2(\Gamma_{11} - \Gamma_{14})}{(\Gamma_{11} + \Gamma_{14})}.
$$

(41)

with the “cross relaxation” rate

$$
\tilde{R}_{1,ISP} = \frac{P_A}{\tau_A} \frac{2(\Gamma_{11} - \Gamma_{14})}{(\Gamma_{11} + \Gamma_{14})}.
$$

(42)
\[
\Gamma_{zz}^{1f} = R_{I,S,P}^{a} \begin{bmatrix}
R_{SS}^{a} + X_{SS}^{a} + i \omega_{0} Q \\
R_{SP}^{a} + X_{SP}^{a} + i \omega_{0} Q \\
R_{PP}^{a} + X_{PP}^{a} + i \omega_{0} Q
\end{bmatrix}^{-1} R_{I,S,P}^{a*},
\]

where \( Q = \text{diag}(0,1,-1) \) and \( R_{I,S,P}^{a} = [R_{I,S}^{a}, R_{I,P}^{a}] \). The elements of the \( 3 \times 3 \) cross-spin relaxation matrix \( R_{SS}^{a} \) and \( R_{PP}^{a} \) are given in Eq. (E.25) and the elements of the \( 3 \times 3 \) auto-spin relaxation matrices \( R_{SS}^{a} \) and \( R_{PP}^{a} \) can be obtained from Eqs. (E.21)-(E.24). All these rates connect single-spin modes and therefore only involve self-correlations.

The static \( SP \) dipole coupling affects the ILRR via the \( 3 \times 3 \) matrices \( X_{SS}^{a}, X_{SP}^{a}, X_{PS}^{a}, \) and \( X_{PP}^{a} \) in the single-spin \( S \) and \( P \) subspace. In Appendix F, \( X \) we show that

\[
X^{a} = \begin{bmatrix}
X_{SS}^{a} & X_{SP}^{a} \\
X_{PS}^{a} & X_{PP}^{a}
\end{bmatrix} = \begin{bmatrix}
A_{S,0}^{a} & 0 \\
0 & A_{P,0}^{a}
\end{bmatrix} \begin{bmatrix}
R_{SS}^{a} + i \omega_{0} Q_{ss0} \\
R_{PP}^{a} + i \omega_{0} Q_{pp0}
\end{bmatrix}^{-1} \begin{bmatrix}
A_{S,0}^{a\dagger} & 0 \\
0 & A_{P,0}^{a\dagger}
\end{bmatrix},
\]

where \( Q_{ss0} = \text{diag}(0,0,0,1,1,-1,-1,2,-2) \). The elements of the \( 9 \times 9 \) relaxation matrix \( R_{SS}^{a} \) in the two-spin-\( SP \) subspace, obtained from Eqs. (E.9) to (E.11), \( \delta I \) involve \( IS \) and \( IP \) two-spin correlations as well as \( IS-I P \) distinct correlations. The elements of the \( 3 \times 9 \) static dipolar Liouvillian matrices \( A_{S,0}^{a} \) and \( A_{P,0}^{a} \) can be obtained from Eq. (E.26).

If we neglect the static \( SP \) dipole coupling, then Eq. (42) reduces to (Appendix F)

\[
\Gamma_{zz}^{1f} = R_{I,S}^{a} \begin{bmatrix}
R_{SS}^{a} + i \omega_{0} Q_{ss0} \\
R_{PS}^{a} + i \omega_{0} Q_{ps0}
\end{bmatrix}^{-1} R_{I,P}^{a*},
\]

showing, with Eq. (41), that the \( IS \) and \( IP \) dipole couplings contribute additively so that \( \Gamma_{zz}^{1f}(IS-I P) = \Gamma_{zz}^{1f}(IS-I) + \Gamma_{zz}^{1f}(IP-I) \), with the two-spin ILRRs as given in Paper II. If spin \( P \) is located far away from spins \( I \) and \( S \), so that both dipole couplings to spin \( P \) are negligibly weak, then the \( IS-I P \) result reduces further to the previously obtained \( \delta I \) result (specialized to isochronous spins) for the asymmetric two-spin case \( IS-I \).

If we (artificially) neglect cross-spin relaxation, so that \( R_{I,S}^{a} = R_{I,P}^{a} = 0 \), then Eqs. (41) and (42) yield

\[
\tilde{R}_{I,I}^{a} = P_{A} \langle \tilde{R}_{zz}^{1f} \rangle = \frac{1}{2\hbar} P_{A} (\omega_{D,IS} + \omega_{D,IP})
\]

\[
\times \left[ j(0) + 3 j(\omega_{0}) + 6 j(2\omega_{0}) \right],
\]

where Eq. (E.20a) was also used. This (unphysical) result shows that cross-spin relaxation is necessary for the ILRR to approach zero at high field.

In the secular approximation, valid when \( \omega_{0} \gg \omega_{D,TA} \) for the fluctuating \( IS \) and \( IP \) dipole couplings and \( \omega_{0} \gg \omega_{IP} \) for the static \( SP \) coupling, Eq. (42) reduces to (Appendix F)

\[
\Gamma_{zz}^{1f} = \frac{R_{SS}^{a} (R_{PP}^{a})^2 + R_{PS}^{a} (R_{IS}^{a})^2 + X(R_{IS}^{a} + R_{PP}^{a})^2}{R_{SS}^{a} (R_{PP}^{a})^2 + X(R_{SS}^{a} + R_{PP}^{a})^2},
\]

with

\[
X = \frac{2}{9} \left[ \omega_{D,SP} D_{zz}^{2} (Q_{SP}) \right] \rho.
\]

Here, \( \rho \) are the “33 element” (corresponding to basis operator \( B_{33} \) in Table S1 A), of the inverse of the \( 3 \times 3 \) \( Q = 0 \) block of \( R \) in the two-spin-\( SP \) subspace. This result shows that, even in the secular approximation, the dipole couplings of the labile spin \( I \) with the two nonlabile spins \( S \) and \( P \) do not contribute independently to the ILRR if \( \omega_{D,SP} \neq 0 \).

### D. Chemical shifts

The BWR results presented in Secs. III A–III C were derived under the assumption that the three spins are isochronous. In Appendix G, \( \delta I \) we generalize the BWR treatment to include the effects of chemical shifts in the Zeeman Hamiltonians \( H_{Z}^{A} \) and \( H_{Z}^{B} \), as shown in Eq. (1). Whereas the SSE theory of Sec. II is valid for arbitrarily large chemical shifts (including heteronuclear spins), we restrict the generalized BWR treatment to homonuclear spins, so that \( \delta \ll 1 \). For example, proton shifts rarely exceed 10 ppm.

Chemical shift effects on the ILRR in the dilute regime are examined in detail in Appendix G. \( \delta I \) One effect is to displace the primary and secondary dispersion steps to higher or lower frequency, but the relative displacement, of order \( \delta \), is negligible for homonuclear spins. For the asymmetric exchange cases \( ISP-I S \) and \( ISP-I P \), this is the only effect of chemical shifts in the MN regime. For the two-spin cases \( IS-I S \) and \( IS-I P \), this is also the (negligibly small) effect of chemical shifts.\( \delta I \)

In contrast, for the symmetric exchange case \( ISP-I S \), chemical shifts can significantly alter the ILRR dispersion profile even if \( \delta \ll 1 \). Formally, this effect can be described by replacing Eq. (33) with (Appendix G)

\[
\langle \rho \rangle_{L} = [R_{L} - P_{A} R_{L/ZQC}(P_{A} R_{ZQC} + i W)^{-1} R_{ZQC/L}]^{-1},
\]

where we have introduced the \( 7 \times 7 \) frequency matrix

\[
W = \omega_{0}(P_{A} D_{A} + P_{B} D_{B}).
\]

The elements of the matrix \( D_{A} \) are linear combinations of the chemical shifts \( \delta_{A}^{a} \) and \( \delta_{P}^{a} \), and similarly for \( D_{B} \), as shown by Eqs. (G.4) and (G.5).

This modification gives rise to a novel secondary dispersion step, not present for isochronous spins, centered at the nonsecular decoupling (NSD) frequency (Appendix G)

\[
\omega_{NSD} = \frac{P_{A} \omega_{D}^{a} \tau_{A}}{P_{A} \Delta^{a} + P_{B} \Delta^{B}},
\]

where \( \Delta^{a} \) can be approximately identified as the largest of \( \delta_{A}^{a} \) and \( \delta_{P}^{a} \), and similarly for \( \Delta^{B} \). At low frequencies, such that \( \omega_{0} \ll \omega_{NSD} \), chemical shifts have no effect on the ILRR. At \( \omega_{0} \approx \omega_{NSD} \), there is an inverted secondary dispersion step as the (negative) contribution from distinct correlations in the L/ZQC cross-mode rates is partly lost. At higher frequencies, such that \( \omega_{0} \gg \omega_{NSD} \), \( \Gamma_{zz}^{1f} \) remains larger for isochronous spins but it never exceeds the ILRR produced solely by self-correlations.

If chemical shifts are present only in state A, so that \( \delta^{B} = 0 \), then Eq. (50) yields \( \omega_{NSD} = \omega_{D}^{a} \tau_{A} / \Delta^{A} \). This result
is also obtained for the special case $P_A = 1$, which has been discussed previously.\(^7\) If the chemical shifts are the same in states $A$ and $B$ ($\delta^A = \delta^B \equiv \delta$), then Eq. (50) yields $\omega_{NSD} = P_A \omega^0_3 \tau_A / \delta$, so the secondary dispersion appears at a much lower frequency. In general, the chemical shift effect can be neglected if the secondary dispersion step occurs well above the primary dispersion, that is, if $\omega_{NSD} \gg 1 / \tau_A$. As seen from Eq. (50), this is true if the chemical shifts in both states are sufficiently small that $|P_A \delta^A + P_B \delta^B| \ll P_A (\omega_D \tau_A)^2$.

IV. NUMERICAL RESULTS

In this section, we present numerical calculations that illustrate the theoretical results obtained in Secs. II and III. Since the prime application is water-$^1$H relaxation in tissue-like systems, we consider only homonuclear spins ($\kappa_S = \kappa_P = 1$) although the SLE theory in Sec. II can also handle heteronuclear spin systems. Moreover, in Secs. IV A–IV C, we assume that the spins are effectively isochronous, so that we can set $\delta_S = \delta_P = 0$ in both states $A$ and $B$. This assumption is justified in Sec. IV D, where we show that the effect of chemical shifts is negligibly small in virtually all situations of practical interest.

In most calculations, the bound fraction of labile spin(s) is set to the experimentally relevant value of $P_A = 10^{-3}$, so we are in the dilute regime. Whereas the BWR theory of Sec. III is restricted to the dilute regime, the SLE theory of Sec. II is valid for arbitrary $P_A$, including the opposite limit of $P_A = 1$. In this limit, Eqs. (12b), (13), (14), and (D.15)\(^7\) yield for the ISP–ISP case

$$
\tilde{R}_{ISP} = 3 \left\{ \sum_{n=1}^{3} \sum_{\rho=1}^{3} \left( \begin{array}{c} 3 \\ n \end{array} \right) \left( \begin{array}{c} 3 \\ \rho \end{array} \right) \left( \frac{1}{\tau_A} + i \mb{L}_Z + i \mb{L}_D \right)^{-1} - 1 \right\}^{-1}, 
$$

(51)

a result that we have used in a previous study.\(^7\) In some calculations, we use $P_A = 10^{-4}$ to ensure that the BWR and SLE results agree to a high level of accuracy.

The calculation of the ILRR from SLE theory proceeds as follows. First, we construct the $63 \times 63$ supermatrix within square brackets in Eq. (14) from the Zeeman and dipolar Liouvillian supermatrices $\mb{L}_Z$ and $\mb{L}_D$ (Appendixes B and C) and the exchange supermatrix $\mb{K}$ (Sec. II A). We then insert this matrix and compute the isotropic average over the Euler angles $\Omega^\alpha \equiv (\psi^\alpha, \theta^\alpha, \varphi^\alpha)$. For the angles $\psi^\alpha$ and $\theta^\alpha$, which determine the orientation of the internuclear vector $\mb{r}^\alpha_{IS}$, we use Lebedev quadrature with 350 points on the unit sphere.\(^{15,16}\) For the angle $\varphi^\alpha$, which determines the orientation of the nuclear plane about the $\mb{r}^\alpha_{IS}$ axis, we use a uniform grid with 30 points. From the $\mb{G}^A(0)$ supermatrix obtained in this way and the exchange matrix $\mb{T}$ (Sec. II A), we obtain $\mb{U}^{BB}(0)$ from Eq. (12a). Finally, the ILRR in the dilute regime for the three exchange cases, $\tilde{R}^\alpha_{ISP}$, $\tilde{R}^\alpha_{ISP}$, or $\tilde{R}^\alpha_{ISP}$, is calculated from Eqs. (24), (22b), or (23c), respectively. When using Eq. (24) for the ISP–I case, the ILRR is actually obtained directly from $\mb{G}^A(0)$, bypassing the calculation of $\mb{U}^{BB}(0)$.

For the calculation of the ILRR from BWR theory, we use Eqs. (41)–(43) for the ISP–I case, Eqs. (36)–(38) for the ISP–IS case and Eqs. (31) and (33) for the ISP–ISP case, along with the required elements of the local relaxation supermatrix $\mb{R}^\alpha$ from Appendix E.\(^8\) The isotropic orientational averages are computed in the same way as for the SLE theory. For all three exchange cases, we have confirmed that the BWR results coincide, to within numerical accuracy, with the SLE results in the MN regime.

A. Overview of dispersion profiles

To provide an overview of the longitudinal relaxation behavior of the three-spin system, we present in Fig. 2 the complete dispersion profiles of $\tilde{R}^\alpha_{ISP}$, $\tilde{R}^\alpha_{ISP}$, and $\tilde{R}^\alpha_{ISP}$ at four values of the mean survival time $\tau_A$, ranging from the MN regime with $\omega_D \tau_A = 0.01$ to the ultraslow-motion (USM) regime with $\omega_D \tau_A = 100$. These 12 profiles, all calculated from the SLE theory, are shown in the panels in left column of Fig. 2. Here, we assume an equilateral triangle geometry ($\beta_1 = \beta_2 = 60^\circ$) so all three dipole couplings have the same magnitude, taken to be $\omega_D = 1 \times 10^5 \text{ rad s}^{-1}$, which corresponds to an internuclear separation of $r_{IS} = 2.245 \text{ Å}$ for two protons. (Other internuclear geometries are considered in Sec. IV B.)

The panels in right column of Fig. 2 show the corresponding dispersion profiles for a two-spin (IS) system with the same values of $\omega_D$ and $\tau_A$ as in the left-hand panel. These profiles were computed from the two-spin dipolar EMOR theory presented in Papers I and II. The IS–I profile can also be obtained from the three-spin ISP–I result by setting $\beta_P$ to a small value, which corresponds to locating spin $P$ far away from spins $I$ and $S$. Likewise, the IS–IS profile can be obtained from the three-spin ISP–IS result with small $\beta_P$. In both of these cases, the remote spin $P$ is nonlabile.

In contrast, if we make the labile spin $P$ remote in the ISP–ISP case, we obtain a situation where the ILRR is very small because the temporal decay of the observed total longitudinal magnetization of the three spins has an extended tail due to slow relaxation of the weakly dipole-coupled spin $P$. Because the magnetization decay is then strongly non-exponential, it is not adequately described by the ILRR. In any event, this “pathological” scenario is of little practical relevance, partly because of the special nature of the spin system and partly because of our neglect of scalar couplings, which would have a substantial effect in this particular case. This situation cannot occur when the exchanging species contains one (ISP–I) or two (ISP–IS)
spins, which are the cases of primary interest for applications (see Sec. I). For those cases, as well as for the ISP–ISP case with magnetic equivalence (or, at least, three similar dipole couplings), we expect that the decay of the total observed longitudinal magnetization of the labile spin(s) is very nearly exponential in the dilute regime, as shown explicitly in Paper II for the two-spin IS–I case.

1. Motional-narrowing regime

In Fig. 2(a), with $\tau_A = 100$ ns and $\omega_D \tau_A = 0.01$, we are squarely in the MN regime, where the BWR results of Sec. III apply. Let us first compare the symmetric exchange cases ISP–ISP and IS–IS, where the spin system exchanges as an intact unit. The IS–IS profile is given in
Paper I

\[
\tilde{R}^{\text{dil}}_{1,1S}(IS-IS) = \frac{2}{3} P_\Lambda \omega_D^2 \left[ \frac{0.2 \tau_A}{1 + (\omega_0 \tau_A)^2} + \frac{0.8 \tau_A}{1 + (2 \omega_0 \tau_A)^2} \right]. \quad (52)
\]

In the three-spin system, each spin is involved in two equally strong dipole couplings, so one might expect \(\tilde{R}^{\text{dil}}_{1,1S}(ISP-ISP)\) to be precisely a factor 2 larger than \(\tilde{R}^{\text{dil}}_{1,1S}(IS-IS)\). This is true as long as we only take self-correlations into account, that is, \(\tilde{R}^{\text{dil,self}}_{1,1S}(ISP-ISP)\) = 2 \(\tilde{R}^{\text{dil}}_{1,1S}(IS-IS)\). However, correlations between distinct (albeit, here, equally strong) dipole couplings also contribute (negatively) to \(\tilde{R}^{\text{dil}}_{1,1S}(ISP-ISP)\). This contribution happens to be quite small for the equilateral triangle geometry considered in Fig. 2, so \(\tilde{R}^{\text{dil}}_{1,1S}(ISP-ISP)\) is merely 0.74% less than 2 \(\tilde{R}^{\text{dil}}_{1,1S}(IS-IS)\) in the extreme-narrowing (EN) regime. (Depending on \(\omega_0\), the relative difference varies between −0.6% and −0.8%, just as for spherical-top rotation in isotropic fluids. 5)

Consider now the asymmetric exchange cases ISP–IS and IS–I, with one nonlabile spin. In the MN regime, which is also the fast-exchange regime for the EMOR model, relaxation rates that couple labile-spin modes are isotropically averaged, as in the first part of Eq. (37). In contrast, relaxation rates involving one or two nonlabile-spin modes are not exchange averaged, cf. Eq. (38). The local relaxation matrix (not exchange-averaged) has lower than axial symmetry and the Wigner-Eckart theorem does not forbid relaxation coupling of local spin modes with different quantum order \(Q\), which we refer to as cross-mode relaxation. 5 If, as is the case here, the exchanging spin system contains one or two spins, the invariant odd-parity subspace only contains single-spin modes. The only available cross-mode relaxation channel is therefore between the longitudinal and transverse magnetizations of the same or different spins, at least one of which is nonlabile. 5

Single-spin cross-mode relaxation is associated with nonsecular terms in the BWR master equation, that is, terms with \(M^T \neq -M\) in Eq. (4.8). 8 It is therefore eliminated by differential Larmor precession at higher frequencies, a phenomenon that we refer to as nonsecular decoupling. As a consequence, the EN regime is split into two subregimes: 5 the zero-field (ZF) regime with \(\omega_0 \ll \omega_D^2 \tau_A\), and the low-field (LF) regime with \(\omega_D^2 \tau_A \ll \omega_0 \ll 1/\tau_A\). Single-spin cross-mode relaxation is only effective in the ZF regime, where it makes a negative contribution to the ILRR. 5 As a result, the dispersion profile exhibits an inverted secondary dispersion step at \(\omega_0 \approx \omega_D^2 \tau_A\), in addition to the primary dispersion step at \(\omega_0 \approx 1/\tau_A\). The dashed dispersion profiles in Fig. 2(a) were computed from the BWR results in the secular approximation, where cross-mode relaxation is neglected. The secular approximation is evidently not valid in the ZF regime. Comparing the ISP–IS and IS–I profiles in Fig. 2(a), we see that, while the secondary dispersion occurs at the same frequency, the step is much smaller for the ISP–IS case (~4% versus a factor ~2).

In the two-spin IS system, the ILRR is reduced by a factor 5 in the ZF regime and by a factor ~2.5 in the LF regime when the \(S\) spin becomes nonlabile, 5 that is, going from IS–IS to IS–I. In contrast, in the three-spin ISP system, the ILRR is only reduced by ~29% in the ZF regime and by ~26% in the LF regime when the \(P\) spin becomes nonlabile, that is, going from ISP–ISP to ISP–IS. This difference reflects the fact that, for the three-spin system, the labile \(I\) and \(S\) spins are each involved in two dipole couplings, only one of which is fragmented by exchange.

Like the ISP–IS profile, the ISP–I profile in Fig. 2(a) exhibits a secondary dispersion in addition to the primary dispersion at \(\omega_0 \approx 1/\tau_A\). However, the secondary dispersion step now appears at \(\omega_0 \approx \omega_D^0 \tau_A\) (rather than at \(\omega_0 \approx \omega_D^2 \tau_A\)) and it is not inverted. (The irregular “fine-structure” in the secondary dispersion step of the ISP–I profiles in Figures 2(a) and 2(b) is a real feature — not a numerical imperfection.) The origin of the secondary dispersion step in the ISP–I profile is the static dipole coupling between the nonlabile spins \(S\) and \(P\). If we set \(\omega_{0,SP} = 0\) without altering the other two (equal) dipole couplings, then this secondary dispersion step disappears and instead an inverted dispersion step at \(\omega_0 \approx \omega_D^0 \tau_A\) appears (the dashed-dotted curve in Fig. 2(a)). In fact, for \(\omega_{0,SP} = 0\), we have \(\tilde{R}^{\text{dil}}_{1,1S}(ISP-I) = 2 \tilde{R}^{\text{dil}}_{1,1S}(IS-IS)\), as noted in Sec. III C.

In Appendix H 5 we examine in detail how the static dipole coupling removes the inverted secondary dispersion step at \(\omega_0 \approx \omega_D^2 \tau_A\) and instead produces a non-inverted secondary dispersion step at \(\omega_0 \approx \omega_D^0 \tau_A\). To characterize the strength of the static dipole coupling, we introduce the dimensionless parameter

\[
\epsilon \equiv \frac{\omega_{0,SP}}{\omega_D^2 \tau_A}, \quad (53)
\]

where \(\omega_{0,}\) we refer to the static dipole coupling \(\omega_{0,SP}\) as weak if \(\epsilon \ll 1\) and as strong if \(\epsilon \gg 1\). For the equilateral triangle geometry assumed in Fig. 2, where all three dipole couplings are equally strong, Eq. (53) implies that \(\epsilon \gg 1\) in the MN regime \(\omega_{0,SP} \ll 1\). The weak coupling limit \(\epsilon \ll 1\) is only relevant when one of the nonlabile spins is located far from the other two spins, in which case the ISP–I case effectively reduces to the IS–I case (Appendix H 5).

The key to understanding the effect of the static dipole coupling is the singular nature of the matrix \(X\) in Eq. (43). Specifically, at all frequencies \(\omega_0\), \(X\) has one zero eigenvalue and the associated eigenvector defines a one-dimensional unitary subspace, even though \(X\) is Hermitian only for \(\omega_0 = 0\) (Appendix H 5). A weak static dipole coupling has no effect at all; the dispersion profile is the same as for \(\omega_{0,SP} = 0\) (the dashed-dotted profile in Fig. 2(a)), with an inverted secondary dispersion at \(\omega_0 \approx \omega_D^2 \tau_A\), where single-spin cross-mode relaxation is abolished by nonsecular decoupling. A strong static dipole coupling projects the cross-spin and auto-spin relaxation matrices onto the unitary subspace associated with the zero eigenvalue (Appendix H 5). As a result, the ZF regime is extended from \(\omega_0 \approx \omega_D^2 \tau_A\) to \(\omega_0 \approx \omega_{0,SP}\), where a secondary dispersion step appears. Above this frequency, single-spin cross-mode relaxation is no longer effective and the secular approximation, Eq. (46), is valid (dashed profile in Fig. 2(a)). Single-spin cross-mode relaxation occurs throughout the extended ZF regime (up to \(\omega_0 \approx \omega_{0,SP}\)), but it is modified by the static dipole coupling. Above \(\omega_0 \approx \omega_{0,SP}\), this modified single-spin cross-mode relaxation is abolished.
by nonsecular decoupling, but this effect, which increases $\tilde{R}_{1,IS}^{dil}$, is overshadowed by the effects of nonsecular decoupling on the matrix $X$ in Eq. (43), which decreases $\tilde{R}_{1,IS}^{dil}$, thus accounting for the non-inverted shape of the secondary dispersion step in the presence of a strong static dipole coupling. Nonsecular decoupling modifies $X$ in two ways. First, it eliminates the nonsecular part of static dipole coupling, that is, terms with $M \neq 0$ in Eq. (E.8). Second, it eliminates the nonsecular part of the two-spin relaxation matrix $R_{1,IS}^{dil}$ corresponding to terms with $M' \neq -M$ in Eq. (E.4). Note also that, although relaxation coupling of two-spin modes with different quantum order $Q$ is thus eliminated, cross-mode relaxation within the $Q$-blocks can still occur above the secondary dispersion ($\omega_0 > \omega_{D,SP}$). Note also that, whereas the strength of the static dipole coupling determines the frequency of the secondary dispersion step, it has no effect on $\tilde{R}_{1,IS}^{dil}$ in the ZF regime as long as the coupling is strong ($\epsilon \gg 1$).

2. Ultraslow-motion regime

Increasing $\tau_A$, thereby moving from the MN regime to the USM regime, has two principal effects on the dispersion profile, as described in Papers I and II for the two-spin system. First, the position of the primary dispersion step at $\omega_0 \approx 1/\tau_A$ is down-shifted in frequency until $\tau_A$ becomes comparable to $1/\omega_D (=10^{-5}$ s, here) and eventually stops at $\omega_0 \approx \omega_D$ when the USM limit is reached. Second, the ILRR in the ZF regime first increases and then decreases, with a maximum near $\omega_D \tau_A \approx 1$. As seen from Figs. 2(a)-2(d), this is true for all three exchange cases. The continuous variation of $\tilde{R}_{1,IS}^{dil}(0)$ with $\tau_A$ is shown in Fig. 3 for the five cases. For the two-spin cases, these curves are given by the analytical ZF results in Papers I and II,

$$\tilde{R}_{1,IS}^{dil}(0) = \frac{2}{3} P_A \frac{\omega_D^2 \tau_A}{[5 + (\omega_D \tau_A)^2]} (IS-IS),$$

and the shape and position of the dispersion profile no longer change as $\tau_A$ is further increased. For the $IS-IS$ case in the USM regime, we showed in Paper I that

$$\tilde{R}_{1,IS}^{dil}(IS-IS) = \frac{2}{3} P_A \frac{0.2}{\tau_A} \left[\frac{0.2}{1 + (\omega_0/\omega_D)^2} + \frac{0.8}{1 + (2 \omega_0/\omega_D)^2}\right],$$

so the dispersion has the same “Lorentzian” shape as in the MN regime, but now with $1/\omega_D$ playing the role of an apparent correlation time. In fact, Eq. (55) is obtained from Eq. (52) by substituting $1/\omega_D$ for $\tau_A$. For the other four cases, where analytical results are not available, the USM profiles are no longer “Lorentzian” but exhibit a “fine-structure” that is particularly striking for the three-spin cases, with two distinct maxima (or “bumps”) for the equilateral triangle geometry (Fig. 2(d)).

B. Internuclear geometry

The ILRR depends on the fixed relative orientation of the internuclear vectors as well as on the length of those vectors, which determines the magnitude of the dipole couplings. In other words, the ILRR depends on the angles $\beta_I$ and $\beta_S$ (Fig. 1) directly via Eq. (4), as well as indirectly via Eq. (6). This is true for all three exchange cases. For the $ISP-IS$ case in the MN regime, Eq. (E.17) shows that the direct dependence on these angles enters solely via the distinct correlations.

So far, we have examined the equilateral triangle geometry, where the three dipole couplings are of equal magnitude. To illustrate the effect of the internuclear geometry, we show in Fig. 4 the dispersion profiles of $\tilde{R}_{1,ISP}^{dil}(ISP-ISP)$, $\tilde{R}_{1,ISP}^{dil}(ISP-IS)$, and $\tilde{R}_{1,ISP}^{dil}(ISP-ISP)$ for five less symmetric nuclear configurations. For all these geometries, the $I-S$ dipole coupling is taken to be the same, $\omega_{D,IS} = 1 \times 10^5$ rad s$^{-1}$. The other two dipole couplings are then determined by the angles $\beta_I$ and $\beta_S$ in Eq. (6). The examined internuclear geometries are drawn to scale in Fig. 4, where thick and dashed-dotted...
FIG. 4. ILRR dispersion profiles for the three-spin cases with different nuclear geometries as indicated to the right. Parameter values: $P_A = 10^{-3}$, $\omega_{D,IS} = 10^5$ rad s$^{-1}$, and $\tau_A = 10^{-6}$ s (left column) or $10^{-4}$ s (right column).

lines represent dipole couplings that are stronger or weaker, respectively, than $\omega_{D,IS}$. For reference, we show the equilateral triangle geometry ($\beta_I = \beta_S = 60^\circ$) in Fig. 4(a). The other panels show the effect of weakening one (Fig. 4(b)) or two (Fig. 4(c)) dipole couplings, or making one coupling stronger and the other weaker (Fig. 4(d)). In Figs. 4(b) and 4(d), two geometries are included that differ by interchange of the $I$ and $S$ spins. Because of the nuclear permutation symmetry of the ILRR (Sec. II C), the $\tilde{R}_{1,IS}^{\text{dil}}$ and $\tilde{R}_{1,ISP}^{\text{dil}}$ profiles are unaffected by this permutation. For the permuted geometry (drawn below the original geometry in Fig. 4), we therefore display only the $\tilde{R}_{1,I}^{\text{dil}}$ profile (dashed).

The mean survival time $\tau_A = 1 \mu$s for the panels in the left column of Fig. 4, whereas $\tau_A = 100 \mu$s for the panels in the right column. These $\tau_A$ values are in the range typically found for internal water molecules and labile protons in globular proteins.
proteins. Because $\omega_{D,IS} \tau_A = 0.1$ in the left column, these dispersion profiles can be rationalized almost quantitatively with the aid of BWR theory (Sec. III). On the other hand, since $\omega_{D,IS} \tau_A = 10$ in the right column, these dispersion profiles are nearly in the USM limit with respect to the $I-S$ dipole coupling, but not always for the other dipole couplings.

We examine first how the dispersion profile is affected by internuclear geometry when $\tau_A = 1 \mu$s (left column of panels). Going from an equilateral triangle (Fig. 4(a)) to an isoceles triangle with $\beta_1 = 140^\circ$ and $\beta_2 = 20^\circ$ (solid curves in Fig. 4(b), corresponding to the uppermost geometry depicted on the right), so $\omega_{D,SP}$ is $\sim 15\%$ of $\omega_{D,IS} = \omega_{D,IP}$, we find that $\tilde{R}^{\text{dil}}_{1,IS}(0)$ and $\tilde{R}^{\text{dil}}_{1,SP}(0)$ are reduced by factors of 0.51 and 0.78, whereas the dispersion shape is nearly unaffected. A twofold reduction of $\tilde{R}^{\text{dil}}_{1,IS}(0)$ is expected in the MN regime if distinct correlations can be neglected, so the three longitudinal magnetizations do not couple with the seven-three spin-zero quantum coherences in $(\mathbf{R}^5)$ of Eq. (32), and if $\omega_{D,SP} = 0$, so $R^{\text{SS}}_{zz}$ and $R^{\text{zz}}_{zz}$ are reduced by a factor 2 and $R^{\text{zz}}_{zz} = 0$ (Appendix E).

In contrast, while $\tilde{R}^{\text{dil}}_{1,IP}(0)$ is only reduced by a factor 0.90, the dispersion shape changes markedly when the static $S-P$ coupling is weakened (Fig. 4(b)). In Fig. 4(a), the secondary dispersion step at $\omega_0 = \omega_{D,SP} = 10^7$ rad s$^{-1}$ merges smoothly with the primary dispersion step at $\omega_0 \approx 1/\tau_A = 10^6$ rad s$^{-1}$. In Fig. 4(b), where $\omega_{D,SP} \approx 1.5 \times 10^5$ rad s$^{-1}$, the secondary dispersion is downshifted and reduced in magnitude, whereas the primary dispersion step is nearly unaffected since the dipole couplings involving the labile I spin are the same as in Fig. 4(a). Interchanging the $I$ and $S$ spins, so that $\beta_1 = 20^\circ$ and $\beta_2 = 140^\circ$, has no effect on the $R^{\text{dil}}_{1,ISP}$ and $\tilde{R}^{\text{dil}}_{1,ISP}$ profiles (Sec. II C), but the $\tilde{R}^{\text{dil}}_{1,IP}$ profile (dashed) changes qualitatively because now the weakened dipole coupling is not the static one.

Next, we consider the isoceles triangle geometry ($\beta_1 = \beta_2 = 75^\circ$) in Fig. 4(c), where $\omega_{D,IP} = \omega_{D,SP}$ is $\sim14$% of $\omega_{D,IS}$. The $\tilde{R}^{\text{dil}}_{1,ISP}(\omega_0)$ profile is strongly affected; relative to Fig. 4(a), $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ is reduced by a factor 0.028, a much larger effect than the factor 1/3 expected (for the self-corrrelations) from removal of two out of three dipole couplings. As already noted, this large reduction of the ILRR is the result of a distinctly biphasic magnetization decay with very slow magnetization transfer from the weakly dipole-coupled spin $P$. In contrast, making the (nonlabile) $P$ spin remote converts the ISP–IS case to the IS–IS case. Indeed, $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ in Fig. 4(c) differs by only 0.8 % from $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ in Fig. 2(b). The ISP–I case, the situation is more complex. Whereas $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ closely follows $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ for $\omega_0 > 10^7$ rad s$^{-1}$, it does not exhibit the pronounced inverted dispersion step seen in Fig. 2(b). Instead, there is a small bump in the $\tilde{R}^{\text{dil}}_{1,ISP}$ profile due to two small overlapping secondary dispersion steps, one of which is inverted. For this geometry, $\epsilon \approx 1.4$ so the static dipole coupling is neither weak nor strong and the secondary dispersion steps appear at almost the same frequency $\omega_0 \approx \omega_{D,SP} \approx \omega_{D,IS}^2 \tau_A$.

Figure 4(d) shows the dispersion profiles for a right-angled triangle geometry with $\beta_1 = 90^\circ$ and $\beta_2 = 30^\circ$, making $\omega_{D,IP}$ larger by a factor 5.2 and $\omega_{D,SP}$ smaller by a factor 0.65 as compared to $\omega_{D,IS}$. The relaxation is now dominated by the strongest dipole coupling $\omega_{D,IP}$, which corresponds to a proton-proton separation of 1.3 Å (less than the smallest physically realized proton-proton separation of $\sim 1.5 \AA$). In contrast to the equilateral triangle geometry in Fig. 4(a), distinct correlations now play an important role for the ISP–ISP profile, reducing $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ by 40% in the BWR approximation and strongly coupling the longitudinal magnetizations to the three-spin zero-quantum coherences. As a result, even though the longitudinal auto-mode rates are now much larger (by a factor 14 for $R^{ISP}_{zz}$ and $R^{ISP}_{zz}$, and by a factor 27 for $R^{ISP}_{zz}$, $\tilde{R}^{ISP}_{1,ISP}(0)$ is actually somewhat smaller than in Fig. 2(b). In addition, the primary dispersion is slightly upshifted and features a small high-frequency shoulder (hardly visible). In Fig. 4(d) (in contrast to Fig. 4(c)), the $\tilde{R}^{dil}_{1,ISP}$ profile happens to be very similar to the $\tilde{R}^{dil}_{1,ISP}$ profile.

Consider now the $\tilde{R}^{\text{dil}}_{1,IP}$ profiles in Fig. 4(d). The upper geometry corresponds to a weak static dipole coupling ($\epsilon = 0.23$), so we observe an inverted secondary dispersion step at $\omega_0 \approx \omega_{D,IS}^2 \tau_A \approx 3 \times 10^5$ rad s$^{-1}$. The lower geometry corresponds to a strong static dipole coupling ($\epsilon \approx 36$), yielding a non-inverted secondary dispersion step at $\omega_0 \approx \omega_{D,SP} \approx 5 \times 10^5$ rad s$^{-1}$, which is hardly visible in Fig. 4(d) because it overlaps with the primary dispersion and because $\tilde{R}^{\text{dil}}_{1,IP}$ is small (since the fluctuating dipole coupling $\omega_{D,IP}$ is 8-fold smaller than for the upper geometry).

We now consider the effect of internuclear geometry when $\tau_A = 100 \mu$s (right column of panels in Fig. 4). Since we are nearly in the USM limit, a further increase of $\tau_A$ hardly affects the shape of the dispersion profiles. Except for certain special cases (see below), the dispersion shape is much less sensitive to the internuclear geometry in the USM regime than in the MN regime. For example, $\tilde{R}^{\text{dil}}_{1,IP}(0)$ differs by at most a few percent among the geometries in Figs. 4(a) and 4(b). This is also true for $\tilde{R}^{\text{dil}}_{1,IS}$ and $\tilde{R}^{\text{dil}}_{1,ISP}$ in Fig. 4(c), although the primary dispersions are slightly down-shifted in frequency as compared to Figs. 4(a) and 4(b). As for $\tau_A = 1 \mu$s, $\tilde{R}^{\text{dil}}_{1,ISP}(0)$ in Fig. 4(c) is strongly suppressed because of slow magnetization transfer from the weakly coupled $P$ spin. The dispersion profiles in Fig. 4(d) vary considerably in shape. The pronounced fine-structure in the $\tilde{R}^{\text{dil}}_{1,IS}$ profile (also present, albeit to lesser extent, in the $\tilde{R}^{\text{dil}}_{1,ISP}$ profile) is likely related to the (orientation-dependent) eigenfrequencies of the dipolar Liouvillian. If the unphysically strong $I-P$ coupling is made slightly weaker by increasing $\beta_S$ from 30 to 37°, whereby $\omega_{D,IP} \approx 2.34 \times 10^5$ rad s$^{-1}$ corresponding to $\tau_{1\text{II}} = 1.69$ Å, the fine-structure almost disappears (blue dashed-dotted curve in Fig. 4(d)).

C. Odd spectral density function

With the exception of the longitudinal auto-mode rates, such as $R^{SS}_{zz}$ and $R^{SP}_{zz}$, the local relaxation rates involve the odd-valued spectral density function (OSDF) $k(\omega) = \omega T_1(\omega)$ as well as the usual even-valued spectral density function $j(\omega) = \tau_A/[1 + (\omega \tau_A)^2]$ (Appendix E). We have recently shown that, contrary to conventional wisdom, the OSDF can affect the longitudinal relaxation of a three-spin system.
without full nuclear permutation symmetry. It is therefore of some theoretical interest to examine the effect of the OSDF in the three EMOR cases: ISP–I, ISP–IS, and ISP–ISP. Because the spectral density function only appears in the BWR theory, this issue is only relevant in the MN regime.

Figure 5 shows the dispersion of $\tilde{R}_{1,ISP}^\text{dil}$ (ISP–ISP) for an isosceles triangle geometry with $\beta_l = 110^\circ$ and $\beta_S = 35^\circ$. The profiles computed from the SLE and BWR theories coincide, as expected since $\omega_D,ISP \tau_A = \omega_D,SP \tau_A = 0.01$ and $\omega_D,SP \tau_A \approx 0.002$, so all three dipole couplings are in the MN regime. The upper panel of Fig. 5 shows that, for this internuclear geometry, removal of the OSDF decreases $\tilde{R}_{1,ISP}^\text{dil}$ by up to 2.5%. Two conditions must be met for the OSDF to influence the longitudinal relaxation of a three-spin system. First, the three spins must be geometrically or dynamically nonequivalent. The three spins are dynamically equivalent in the ISP–ISP EMOR model, but they are not geometrically equivalent for the geometry considered in Fig. 5. (The OSDF has no effect on $\tilde{R}_{1,ISP}^\text{dil}$ for the equilateral geometry examined in Fig. 2.) Second, the OSDF can only have an effect in the dispersive regime, $0.1 \leq \omega_0 \tau_A \leq 10$, as is evident from Fig. 5. For $\tilde{R}_{1,ISP}^\text{dil}$, which is governed by the $10 \times 10$ odd-parity zero-quantum block of the isotropically averaged relaxation supermatrix $\langle \mathbf{R}^a \rangle$ in Eq. (31), the OSDF only appears in the cross-mode rates that couple the seven odd-rank modes with the three even-rank modes. If the OSDF is neglected, it is therefore sufficient in Eq. (31) to invert the odd-rank $7 \times 7$ block. Moreover, because the coupling of the three longitudinal modes with the seven zero-quantum coherences is mediated entirely by distinct correlations, the OSDF can only affect $\tilde{R}_{1,ISP}^\text{dil}$ via the distinct correlations. For the geometry considered in Fig. 5, distinct correlations are seen to make a large (negative) contribution to $\tilde{R}_{1,ISP}^\text{dil}$, thereby maximizing the OSDF effect.

For the ISP–IS case, the OSDF has no effect at all on $\tilde{R}_{1,ISP}^\text{dil}$. To demonstrate this, we first note that, if there is an OSDF effect, it must be fully manifested already in the secular approximation, valid for $\omega_0 \tau_A \gg (\omega_D,\tau_A)^2$. This follows because $(\omega_D,\tau_A)^2 \ll 1$ in the MN regime and $\omega_0 \tau_A \sim 1$ in the dispersive regime. Any OSDF effect on $\tilde{R}_{1,ISP}^\text{dil}$ must therefore be contained in Eq. (39). As seen from Eq. (40), this expression only involves longitudinal auto-mode rates, which are unaffected by the OSDF (Appendix E). Hence, the OSDF cannot affect $\tilde{R}_{1,ISP}^\text{dil}$ as we have also confirmed numerically.

For the ISP–I case, the same arguments imply that the OSDF can only affect $\tilde{R}_{1,ISP}^\text{dil}$ via the “cross-relaxation” rate $\Gamma_{zz}^I$ in Eq. (46). Since the longitudinal auto-mode rates are unaffected by the OSDF (Appendix E), any OSDF effect must enter via the quantity $X$ in Eq. (47), specifically via the relaxation rates in the $3 \times 3$ zero-quantum two-spin-SP relaxation matrix. Figure 6 shows the dispersion of $\tilde{R}_{1,ISP}^\text{dil}$ for an equilateral triangle geometry. The profiles computed from the SLE and BWR theories coincide, as expected since $\omega_D,\tau_A = 0.01$. As seen from the upper panel (red solid curve), omission of the OSDF increases $\tilde{R}_{1,ISP}^\text{dil}$ in the dispersive regime, but only by a tiny amount (at most 14 ppm in this example). To demonstrate that this really is an OSDF effect, we multiplied $k(\omega)$ by a factor of 100. The effect of removing this artificially inflated OSDF (blue dashed-dotted curve in Fig. 6) is qualitatively the same, but more than three orders of magnitude larger than for the true OSDF. The opposite signs of the OSDF effects in Figs. 5 and 6 reflect their different origins: via cross-mode relaxation between odd-rank and even-rank zero-quantum modes for the ISP–ISP case, and via the static dipole coupling $\omega_D,SP$ and two-spin rates for the ISP–I case. We note also that the OSDF can influence $\tilde{R}_{1,ISP}^\text{dil}$ even when the three spins are geometrically equivalent, because they are not dynamically equivalent in the ISP–I case.

D. Chemical shifts

For the ISP-ISP case and in the MN regime, chemical shifts increase the ILRR above the frequency, $\omega_\text{NSD}$, where cross-mode relaxation is largely eliminated (Sec. III D). This behavior is illustrated in Fig. 7 for a nuclear configuration with $\beta_l = 90^\circ$ and $\beta_S = 30^\circ$. For the chosen parameter values, $\omega_D,X \tau_A \leq 0.005$, so all three dipole couplings are in the MN regime. As expected, the profiles computed from
the generalized BWR theory (Sec. III D, Appendix G)\(^8\) coincide with the profiles computed from the SLE theory, which incorporates the full effect of chemical shifts. For the internuclear geometry examined in Fig. 7, cross-mode relaxation is almost completely abolished even for modest shifts, so that, for \(\omega_0 \gg \omega_{\text{NSD}}, \bar{R}_{\text{ILRR}}^{\text{1,ISP}}\) nearly coincides with the ILRR \(\bar{R}_{\text{ILRR}}^{\text{1,ISP}}\) induced solely by self-correlations.

This is true for most other geometries, but for the equilateral triangle geometry, chemical shifts only eliminate 60\% of the cross-mode contribution to \(\bar{R}_{\text{ILRR}}^{\text{1,ISP}}\). (Due to the small cross-mode contribution for that geometry, the shift effect is merely 0.4\%, other parameters being the same as in Fig. 7.)

For the dispersion profile in Fig. 7 pertaining to equal shifts in states A and B, the secondary dispersion step exhibits two substeps, corresponding to NSD frequencies of \(\sim 10^3\) and \(\sim 2 \times 10^4\) rad s\(^{-1}\), as predicted by Eq. (50) for the \(I-S\) and \(I-P\) dipole couplings, respectively, adopted here. Also in accordance with Eq. (50), the secondary dispersion is upshifted by a factor \(1/P_\Lambda = 10^4\) when the shifts are removed from state B (Fig. 7).

Within the MN regime, the chemical shift effect on the \(\bar{R}_{\text{ILRR}}^{\text{1,ISP}}\) dispersion profile becomes negligibly small when \(\omega_{\text{NSD}} \tau_{\Lambda} \gg 1\), or, for equal shifts in states A and B, when \(\delta \ll P_\Lambda(\omega_0 \tau_{\Lambda})^2\) (Sec. III D). For \(P_\Lambda = 10^{-3}\), \(\omega_0 \approx 10^5\) rad s\(^{-1}\), and chemical shifts of order 10 ppm, we thus expect the chemical shift effect to be negligible for \(\tau_{\Lambda} \gg 10^{-6}\) s. Although this prediction is strictly valid only within the MN regime, Fig. 8 shows that it is consistent with the (exact) SLE-based results.

For the dispersion profile in Fig. 8 pertaining to equal shifts in states A and B, the secondary dispersion step exhibits two substeps, corresponding to NSD frequencies of \(\sim 10^3\) and \(\sim 2 \times 10^4\) rad s\(^{-1}\), as predicted by Eq. (50) for the \(I-S\) and \(I-P\) dipole couplings, respectively, adopted here. Also in accordance with Eq. (50), the secondary dispersion is upshifted by a factor \(1/P_\Lambda = 10^4\) when the shifts are removed from state B (Fig. 7).

Within the MN regime, the chemical shift effect on the \(\bar{R}_{\text{ILRR}}^{\text{1,ISP}}\) dispersion profile becomes negligibly small when \(\omega_{\text{NSD}} \tau_{\Lambda} \gg 1\), or, for equal shifts in states A and B, when \(\delta \ll P_\Lambda(\omega_0 \tau_{\Lambda})^2\) (Sec. III D). For \(P_\Lambda = 10^{-3}\), \(\omega_0 \approx 10^5\) rad s\(^{-1}\), and chemical shifts of order 10 ppm, we thus expect the chemical shift effect to be negligible for \(\tau_{\Lambda} \gg 10^{-6}\) s. Although this prediction is strictly valid only within the MN regime, Fig. 8 shows that it is consistent with the (exact) SLE-based results.

For the dispersion profile in Fig. 8 pertaining to equal shifts in states A and B, the secondary dispersion step exhibits two substeps, corresponding to NSD frequencies of \(\sim 10^3\) and \(\sim 2 \times 10^4\) rad s\(^{-1}\), as predicted by Eq. (50) for the \(I-S\) and \(I-P\) dipole couplings, respectively, adopted here. Also in accordance with Eq. (50), the secondary dispersion is upshifted by a factor \(1/P_\Lambda = 10^4\) when the shifts are removed from state B (Fig. 7).

Within the MN regime, the chemical shift effect on the \(\bar{R}_{\text{ILRR}}^{\text{1,ISP}}\) dispersion profile becomes negligibly small when \(\omega_{\text{NSD}} \tau_{\Lambda} \gg 1\), or, for equal shifts in states A and B, when \(\delta \ll P_\Lambda(\omega_0 \tau_{\Lambda})^2\) (Sec. III D). For \(P_\Lambda = 10^{-3}\), \(\omega_0 \approx 10^5\) rad s\(^{-1}\), and chemical shifts of order 10 ppm, we thus expect the chemical shift effect to be negligible for \(\tau_{\Lambda} \gg 10^{-6}\) s. Although this prediction is strictly valid only within the MN regime, Fig. 8 shows that it is consistent with the (exact) SLE-based results.
In Appendix G,\textsuperscript{8} we show that chemical shifts have no significant effect for the asymmetric exchange cases ISP–IS and ISP–I as long as we are in the MN regime. Calculations based on the SLE theory confirm that this is true also outside the MN regime, the relative shift effect being less than 0.01\% in all examined cases.

\section*{V. CONCLUSIONS}

In Papers I and II, the general non-perturbative stochastic theory of longitudinal relaxation by the dipolar EMOR mechanism\textsuperscript{3} was implemented for two-spin systems with symmetric (IS–IS) and asymmetric (IS–I) exchange, respectively. Here, we have implemented the theory for three-spin systems with symmetric (ISP–ISP) or asymmetric (ISP–IS and ISP–I) exchange. The theory is valid for homonuclear as well as heteronuclear spins and for any distribution of labile spins between the anisotropic (A) sites and the isotropic bulk (B) state. However, because water-proton relaxation in tissue-like systems is arguably the most important application of the dipolar EMOR theory, our theoretical analysis emphasizes homonuclear spin systems in the dilute regime ($P_A \ll 1$). Within this realm, the three examined exchange cases might describe a protein-bound H$_2$O$^+$ ion (ISP–ISP), an internal H$_2$O molecule with a nearby nonlabile proton (ISP–IS), or a labile O–H or N–H proton with two nearby nonlabile protons (ISP–I).

A substantial part of the present study concerns the development of a perturbation (BWR) theory for the three-spin dipolar EMOR model. The semi-analytical results obtained for this limiting form of the general (SLE-based) theory reveal explicitly how the various features of the relaxation dispersion profile emerge from the interplay of the Larmor frequency (including chemical shifts), the static dipole coupling (in the ISP–I case), and specific elements of the local relaxation supermatrix, including cross-spin and cross-mode relaxation rates. Apart from the primary dispersion step at $\omega_0 \approx 1/\tau_A$, three kinds of secondary dispersion step can appear. (1) For the ISP–IS case and the ISP–I case with a weak static dipole coupling, an inverted secondary dispersion appears at $\omega_0 \approx \omega_{D,I}^2 \tau_A$ because of nonsecular decoupling of longitudinal-transverse cross-mode relaxation. (2) For the ISP–I case with a strong static dipole coupling, a (non-inverted) secondary dispersion appears at $\omega_0 \approx \omega_{D,SP}^2$, primarily due to decoupling of the nonsecular parts of the static dipole coupling and the two-spin relaxation matrix. (3) For the ISP–ISP case, an inverted secondary dispersion step appears at $\omega_0 \approx \omega_{D,I}^2 \tau_A / \delta^A$ or $P_A \omega_{D,I}^2 \tau_A / \delta$ in the presence of chemical shifts.

Compared to our findings for the two-spin system in Papers I and II, the present analysis of the three-spin system has revealed several new phenomena or features:

\begin{itemize}
  \item Correlations between distinct dipole couplings affect the longitudinal relaxation in the MN regime for the ISP–ISP case (via one-spin/three-spin and three-spin/three-spin rates) and for the ISP–I case (via two-spin/two-spin rates), but not for the ISP–IS case (which only involves one-spin/one-spin rates).
  \item The shape of the longitudinal relaxation dispersion profile depends on the relative orientation of the internuclear vectors, as well as on their lengths. For some internuclear geometries, the dispersion profile exhibits a fine-structure, particularly pronounced in the USM limit.
  \item For the ISP–I case, a strong static S–P dipole coupling gives rise to a secondary dispersion at $\omega_0 \approx \omega_{D,SP}$, while removing the inverted secondary dispersion seen at $\omega_0 \approx \omega_{D,I}^2 \tau_A$ with a weak (or absent) static dipole coupling. A strong static coupling thus extends the zero-field regime up to $\omega_0 \approx \omega_{D,SP}$.
  \item The longitudinal relaxation dispersion profile is significantly affected by chemical shifts only in the ISP–ISP case, where an inverted secondary dispersion step appears when the contribution from cross-mode relaxation is (partly) eliminated. Outside the MN regime, the chemical shift effect gradually disappears with increasing $\tau_A$.
  \item In the region of the primary dispersion step in the MN regime, the OSSF can increase the ILRR by few percent for the ISP–ISP case and decrease it by a few ppm for the ISP–I case. The OSSF is manifested exclusively via distinct correlations and therefore has no effect for the ISP–IS case, which only involves self-correlations.
\end{itemize}

In a forthcoming final part of this series of papers, we will use the insights gained from our detailed studies of the two-spin and three-spin systems to formulate an approximate theory of longitudinal relaxation by the dipolar EMOR mechanism in multi-spin systems with one or two labile spins, which will then be applied to experimental $^1$H relaxation data from aqueous protein gels.

\section*{ACKNOWLEDGMENTS}

This work was financially supported by the Swedish Research Council.

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\bibitem{Chang Halle JCP 2016} Z. Chang and B. Halle, \textit{J. Chem. Phys.} \textbf{144}, 084202 (2016), also referred to as Paper II.
\bibitem{Chemical shifts} See supplementary material at http://dx.doi.org/10.1063/1.4955423 for tables of ISTO basis operators (Appendix A); Zeeman Liouvillian supermatrix elements (Appendix B); dipolar Liouvillian supermatrix elements (Appendix C); general ILRR expressions for all cases.
\end{thebibliography}
Appendix D); the local relaxation supermatrix in the MN regime
(Appendix E); specific ILRR expressions for the the MN regime
(Appendix F); chemical shift effects in the MN regime (Appendix G);
and the effect of the static dipole coupling for the ISP – I case in the MN
regime (Appendix H).

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(1986).
Nuclear magnetic relaxation by the dipolar EMOR mechanism:
Three-spin systems

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POB 124, SE-22100 Lund, Sweden
APPENDIX A: SPIN OPERATOR BASES

As a basis for three-spin Liouville space, we use the 64 irreducible spherical tensor operators (ISTOs) \( T^K_{Q}(k_I k_S \{ \bar{K} \} k_P) \), constructed by two consecutive couplings of the set of four orthonormal single-spin ISTOs for each spin, e.g.,

\[
T^0_0(I) = \frac{1}{\sqrt{2}} E_I; \quad T^1_0(I) = \sqrt{2} I_z; \quad T^1_{\pm 1}(I) = \mp I_{\pm},
\]

(A.1)

to obtain\(^{1,2}\)

\[
T^K_{Q}(k_I k_S \{ \bar{K} \} k_P) = (-1)^{k_I - k_S - k_P + \bar{K} + Q (2\bar{K} + 1)^{1/2}} (2\bar{K} + 1)^{1/2}
\times \sum_{\bar{Q} = -K}^{K} \sum_{q_I = -k_I}^{k_I} (-1)^{\bar{Q}} \begin{pmatrix}
\bar{K} & k_P & K \\
\bar{Q} & Q - \bar{Q} & -Q
\end{pmatrix}
\begin{pmatrix}
k_I & k_S & \bar{K} \\
q_I & Q - q_I & -Q
\end{pmatrix}
\times T^k_I(I) T^{k_S}_S(S) T^{k_P}_P(P),
\]

(A.2)

where \( \bar{K} \) is the rank of the intermediate tensor operator obtained by first coupling spins \( I \) and \( S \). Here, and in the following, the rank superscript is written in upper case for ISTOs that are normalized in three-spin Liouville space and in lower case for ISTOs that are normalized in single-spin Liouville space. The ISTOs \( T^2_M(X) \) appearing in the dipolar Hamiltonian (3) belong to this basis set; e.g., \( T^2_0(IS) = T^2_0(11\{2\}0) = (3I_z S_z - I \cdot S)/\sqrt{3} \).

The 63 basis operators (excluding the identity operator) listed in Tables S1 A – D are used to describe the three-spin system in state A and, in the symmetric case \( ISP - ISP \), also in state B. For the asymmetric case \( ISP - IS \), where state B only contains two spins, we use a basis comprising the 15 operators in Tables S1 A – D with \( k_P = 0 \). For convenience, these 15 operators are collected in Table S2. For the asymmetric case \( ISP - I \), where state B only contains one spin, we use a basis comprising the 3 operators in Tables S1 A – C with \( k_S = k_P = 0 \). For convenience, these 3 operators are collected in Table S3. All the operators in Tables S1 – S3 are normalized in the same three-spin Liouville space. For example,

\[
(B_1 | B_1) = \frac{1}{2} (I_z | I_z) = \frac{1}{2} \text{Tr}\{I_z^2\} = \frac{1}{2} \text{Tr}_I\{I_z^2\} \times \text{Tr}_S\{E_S\} \times \text{Tr}_P\{E_P\} = \frac{1}{2} \times \frac{1}{2} \times 2 \times 2 = 1.
\]
TABLE S1 A. Spin basis operators $B_n = T^K_0(k_Ik_S\{\vec{K}\}k_P)$ for three spins ISP.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K$</th>
<th>$k_I$</th>
<th>$k_S$</th>
<th>$k_P$</th>
<th>$\vec{K}$</th>
<th>$\mathcal{W}^a$</th>
<th>$B_n^{b,c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-$</td>
<td>$\frac{1}{\sqrt{2}} I_z$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-$</td>
<td>$\frac{1}{\sqrt{2}} S_z$</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-$</td>
<td>$\frac{1}{\sqrt{2}} P_z$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>$-$</td>
<td>$-\frac{2\sqrt{6}}{3} (\mathbf{I} \cdot \mathbf{S}) P_z$</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-$</td>
<td>$\sqrt{2} [I_z(\mathbf{S} \cdot \mathbf{P}) - S_z(\mathbf{I} \cdot \mathbf{P})]$</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$\frac{2}{\sqrt{30}} [2 P_z(\mathbf{I} \cdot \mathbf{S}) - 3 I_z(\mathbf{S} \cdot \mathbf{P}) - 3 S_z(\mathbf{I} \cdot \mathbf{P})]$</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$\frac{2}{\sqrt{5}} [5 I_z S_z P_z - I_z(\mathbf{S} \cdot \mathbf{P}) - S_z(\mathbf{I} \cdot \mathbf{P}) - P_z(\mathbf{I} \cdot \mathbf{S})]$</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>$-$</td>
<td>$-i \frac{2}{\sqrt{3}} (\mathbf{I} \times \mathbf{S}) \cdot \mathbf{P}$</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-$</td>
<td>$i \frac{2}{\sqrt{6}} (\mathbf{I} \times \mathbf{S}) \cdot (3 P_z e_z - \mathbf{P})$</td>
</tr>
<tr>
<td>10</td>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$i \sqrt{2} [I_z(\mathbf{S} \times \mathbf{P}) + S_z(\mathbf{I} \times \mathbf{P})] \cdot e_z$</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>$+$</td>
<td>$-\frac{2}{\sqrt{6}} \mathbf{I} \cdot \mathbf{S}$</td>
</tr>
<tr>
<td>12</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$-\frac{2}{\sqrt{6}} \mathbf{I} \cdot \mathbf{P}$</td>
</tr>
<tr>
<td>13</td>
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<td>0</td>
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<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$-\frac{2}{\sqrt{6}} \mathbf{S} \cdot \mathbf{P}$</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{3}} [3 I_z S_z - \mathbf{I} \cdot \mathbf{S}]$</td>
</tr>
<tr>
<td>15</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{3}} [3 I_z P_z - \mathbf{I} \cdot \mathbf{P}]$</td>
</tr>
<tr>
<td>16</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{3}} [3 S_z P_z - \mathbf{S} \cdot \mathbf{P}]$</td>
</tr>
<tr>
<td>17</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>$+$</td>
<td>$i (\mathbf{I} \times \mathbf{S}) \cdot e_z$</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
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<td>$i (\mathbf{I} \times \mathbf{P}) \cdot e_z$</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$i (\mathbf{S} \times \mathbf{P}) \cdot e_z$</td>
</tr>
</tbody>
</table>

$a$ Parity of $B_n$ under spin inversion conjugation. $b$ Identity operators have been omitted.

c $e_z$ denotes the unit vector along the z axis.
### TABLE S1 B. Spin basis operators $B_n = T^K_1(k_I k_S \{K\} k_P)$ for three spins ISP.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K$</th>
<th>$k_I$</th>
<th>$k_S$</th>
<th>$k_P$</th>
<th>$K$</th>
<th>$W^a$</th>
<th>$B_n^b$</th>
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<td>$-\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>21</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>−</td>
<td>$-\frac{1}{2} S_+$</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>−</td>
<td>$-\frac{1}{2} P_+$</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>−</td>
<td>$\frac{2}{\sqrt{3}} (I \cdot S) P_+$</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−</td>
<td>$(I_z S_+ - I_+ S_+) P_+ + \frac{1}{2} (I_- S_+ - I_+ S_-) P_+$</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$\frac{\sqrt{3}}{\sqrt{5}} [(I_z S_+ + I_+ S_+) P_+ + I_+ S_+ P_- - I_+ S_+ I_- S_- P_+]$</td>
</tr>
<tr>
<td>26</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−</td>
<td>$(I_z S_+ - I_+ S_+) P_+ - \frac{1}{2} (I_- S_+ - I_+ S_-) P_+$</td>
</tr>
<tr>
<td>27</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$\frac{\sqrt{3}}{6} [(4 I_z S_z - I_+ S_- - I_- S_+) P_+ + 2 I_+ S_+ P_- - 2 (I_z S_+ + I_+ S_+) P_z]$</td>
</tr>
<tr>
<td>28</td>
<td>3</td>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$-\frac{1}{\sqrt{15}} [(4 I_z S_z - I_+ S_- - I_- S_+) P_+ + I_+ S_+ P_- + 4 (I_z S_+ + I_+ S_+) P_z]$</td>
</tr>
<tr>
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<td>$\frac{1}{\sqrt{2}} (I_z S_+ - I_+ S_z)$</td>
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<td>+</td>
<td>$\frac{1}{\sqrt{2}} (S_z P_+ - S_+ P_z)$</td>
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<tr>
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<td>1</td>
<td>0</td>
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<td>+</td>
<td>$-\frac{1}{\sqrt{2}} (I_z S_+ + I_+ S_z)$</td>
</tr>
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<td>1</td>
<td>1</td>
<td>+</td>
<td>$-\frac{1}{\sqrt{2}} (I_z P_+ + I_+ P_z)$</td>
</tr>
<tr>
<td>34</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>$-\frac{1}{\sqrt{2}} (S_z P_+ + S_+ P_z)$</td>
</tr>
</tbody>
</table>

$^a$ Parity of $B_n$ under spin inversion conjugation. $^b$ Identity operators have been omitted.
**TABLE S1 C.** Spin basis operators $B_n = T^K_{-1}(k_I k_S \{ \bar{K} \} k_P)$ for three spins $ISP$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K$</th>
<th>$k_I$</th>
<th>$k_S$</th>
<th>$k_P$</th>
<th>$\bar{K}$</th>
<th>$W^a$</th>
<th>$B^b_n$</th>
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<td>1</td>
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<td>$-$</td>
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</tr>
<tr>
<td>36</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>$-$</td>
<td>$\frac{1}{2} S_-$</td>
</tr>
<tr>
<td>37</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>$-$</td>
<td>$\frac{1}{2} P_-$</td>
</tr>
<tr>
<td>38</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>$-$</td>
<td>$-\frac{2}{\sqrt{3}} (I \cdot S) P_-$</td>
</tr>
<tr>
<td>39</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-$</td>
<td>$-(I_z S_- - I_+ S_+) P_- + \frac{1}{2} (I_- S_+ + I_+ S_-) P_-$</td>
</tr>
<tr>
<td>40</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$-\frac{3}{\sqrt{5}} [(I_z S_- + I_+ S_+) P_- + I_- S_- P_+ - I_z S_- P_- + \frac{1}{3} (I \cdot S) P_-]$</td>
</tr>
<tr>
<td>41</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$-$</td>
<td>$(I_- S_+ - I_- S_-) P_- + \frac{1}{2} (I_- S_+ - I_+ S_-) P_-$</td>
</tr>
<tr>
<td>42</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$\frac{3}{\sqrt{6}} [(4 I_z S_- - I_+ S_- - I_- S_+) P_- + 2 I_- S_- P_+ - 2 (I_z S_- + I_- S_-) P_-]$</td>
</tr>
<tr>
<td>43</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$-$</td>
<td>$\frac{1}{\sqrt{15}} [(4 I_z S_- - I_+ S_- - I_- S_+) P_- - I_- S_- P_+ + 4 (I_z S_- + I_- S_-) P_-]$</td>
</tr>
<tr>
<td>44</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (I_z S_- - I_- S_-)$</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (I_+ P_- - I_- P_-)$</td>
</tr>
<tr>
<td>46</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (S_+ P_- - S_- P_-)$</td>
</tr>
<tr>
<td>47</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (I_+ S_- + I_- S_-)$</td>
</tr>
<tr>
<td>48</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (I_+ P_- + I_- P_-)$</td>
</tr>
<tr>
<td>49</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+$</td>
<td>$\frac{1}{\sqrt{2}} (S_+ P_- + S_- P_-)$</td>
</tr>
</tbody>
</table>

*a Parity of $B_n$ under spin inversion conjugation. b Identity operators have been omitted.*
**TABLE S1 D.** Spin basis operators $B_n = T_Q^k (k_I k_S \{ \bar{K} \} k_P)$ with $Q = \pm 2$ or $\pm 3$ for three spins $ISP$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q$</th>
<th>$K$</th>
<th>$k_I$</th>
<th>$k_S$</th>
<th>$k_P$</th>
<th>$K$</th>
<th>$W^a$</th>
<th>$B_n^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−</td>
<td>$-(I_z S_+ - I_+ S_z) P_+$</td>
</tr>
<tr>
<td>51</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$\frac{1}{\sqrt{3}} [2 I_+ S_+ P_z - (I_z S_+ + I_+ S_z) P_+]$</td>
</tr>
<tr>
<td>52</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$\frac{\sqrt{6}}{3} [I_+ S_+ P_z + (I_z S_+ + I_+ S_z) P_+]$</td>
</tr>
<tr>
<td>53</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} I_+ S_+$</td>
</tr>
<tr>
<td>54</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} I_+ P_+$</td>
</tr>
<tr>
<td>55</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} S_+ P_+$</td>
</tr>
<tr>
<td>56</td>
<td>−2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−</td>
<td>$(I_z S_- - I_- + S_z) P_-$</td>
</tr>
<tr>
<td>57</td>
<td>−2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$-\frac{1}{\sqrt{3}} [2 I_- S_- P_z - (I_z S_- + I_- + S_z) P_-]$</td>
</tr>
<tr>
<td>58</td>
<td>−2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$\frac{\sqrt{6}}{3} [I_- S_- P_z + (I_z S_- + I_- S_z) P_-]$</td>
</tr>
<tr>
<td>59</td>
<td>−2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} I_- S_-$</td>
</tr>
<tr>
<td>60</td>
<td>−2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} I_- P_-$</td>
</tr>
<tr>
<td>61</td>
<td>−2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>$\frac{1}{\sqrt{2}} S_- P_-$</td>
</tr>
<tr>
<td>62</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$-I_+ S_+ P_+$</td>
</tr>
<tr>
<td>63</td>
<td>−3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>−</td>
<td>$I_- S_- P_-$</td>
</tr>
</tbody>
</table>

$^a$ Parity of $B_n$ under spin inversion conjugation. $^b$ Identity operators have been omitted.
### TABLE S2. Spin basis operators $B_n = T^K_Q(k_I k_S)$ for two spins $I S$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q$</th>
<th>$K$</th>
<th>$k_I$</th>
<th>$k_S$</th>
<th>$W^a$</th>
<th>$B_n^{b,c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-1 \sqrt{2} I_z$</td>
<td>$\frac{1}{\sqrt{2}} I_z$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-1 \sqrt{2} S_z$</td>
<td>$-\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$+\frac{2}{\sqrt{6}} I \cdot S$</td>
<td>$-\frac{1}{2} I_-$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+i (I \times S) \cdot e_z$</td>
<td>$\frac{1}{2} I_-$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$+\frac{1}{\sqrt{3}} (3 I_z S_z - I \cdot S)$</td>
<td>$\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{2} I_+$</td>
<td>$-\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1}{2} S_z$</td>
<td>$-\frac{1}{2} S_z$</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+\frac{1}{\sqrt{2}} (I_z S_+ - I_+ S_z)$</td>
<td>$\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$-\frac{1}{\sqrt{2}} (I_z S_+ + I_+ S_z)$</td>
<td>$\frac{1}{2} I_-$</td>
</tr>
<tr>
<td>10</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{2} I_-$</td>
<td>$-\frac{1}{2} S_-$</td>
</tr>
<tr>
<td>11</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-\frac{1}{2} S_-$</td>
<td>$-\frac{1}{2} S^\dagger$</td>
</tr>
<tr>
<td>12</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$+\frac{1}{\sqrt{2}} (I_z S_- - I_- S_z)$</td>
<td>$\frac{1}{2} S^-_+$</td>
</tr>
<tr>
<td>13</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}} (I_z S_- + I_- S_z)$</td>
<td>$\frac{1}{2} S^\dagger$</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}} I_+ S_+$</td>
<td>$\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>15</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}} I_- S_-$</td>
<td>$\frac{1}{2} I_-$</td>
</tr>
</tbody>
</table>

$^a$ Parity of $B_n$ under spin inversion conjugation. $^b$ Identity operators have been omitted. $^c$ The operators $B_n$ are normalized in the three-spin Liouville space.

### TABLE S3. Spin basis operators $B_n = T^K_Q$ for a single spin $I$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q$</th>
<th>$K$</th>
<th>$B_n^{a,b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}} I_z$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$-\frac{1}{2} I_+$</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>$\frac{1}{2} I_-$</td>
</tr>
</tbody>
</table>

$^a$ Identity operators have been omitted. $^b$ The operators $B_n$ are normalized in the three-spin Liouville space.
**APPENDIX B: ZEEMAN SUPERMATRIX**

Here we calculate the supermatrix representation of the Zeeman Liouvillian $L_Z = [H_Z, \ ]$ in the ISTO basis of Table S1. To this end, we first consider the matrix representation of the superoperator $\mathcal{I}_z = [I_z, \ ]$, that is,

$$
\left( T_Q^K (k_I k_S \{ \bar{K} \} k_P) \right) | \mathcal{I}_z | T_{Q'}^{K'} (k'_I k'_S \{ \bar{K}' \} k' P)
= \sqrt{2} \left[ \left( T_Q^K (k_I k_S \{ \bar{K} \} k_P) | T_0^1 (10(10)| T_{Q'}^{K'} (k'_I k'_S \{ \bar{K}' \} k' P) \right)
- \left( T_Q^K (k_I k_S \{ \bar{K} \} k_P) | T_{Q'}^{K'} (k'_I k'_S \{ \bar{K}' \} k' P) | T_0^1 (10(10)) \right) \right].
$$

Regarding the coupled $I$ and $S$ spins as a composite system, we can write this as

$$
\left( T_Q^K (k_I k_S \{ \bar{K} \} k_P) | \mathcal{I}_z | T_{Q'}^{K'} (k'_I k'_S \{ \bar{K}' \} k' P) \right)
= \sqrt{2} \left[ \left( T_Q^K (\bar{K} k_P) | T_0^1 (10)| T_{Q'}^{K'} (\bar{K}' k' P) \right) - \left( T_Q^K (\bar{K} k_P) | T_{Q'}^{K'} (\bar{K}' k' P) | T_0^1 (10) \right) \right].
$$

According to the Wigner-Eckart theorem,\(^1\)

$$
\left( T_Q^K (\bar{K} k_P) | T_0^1 (10)| T_{Q'}^{K'} (\bar{K}' k' P) \right) = \delta_{QQ'} (-1)^{K-Q} \left( \begin{array}{cc} K & 1 \\ -Q & 0 \end{array} \right) (B.1)
\times (2K + 1)^{1/2} \left( T^K (\bar{K} k_P) || T^1 (10) || T^{K'} (\bar{K}' k' P) \right),
$$

and

$$
\left( T_Q^K (\bar{K} k_P) | T_{Q'}^{K'} (\bar{K}' k' P) | T_0^1 (10) \right) = \delta_{QQ'} (-1)^{K-Q} \left( \begin{array}{cc} K & 1 \\ -Q & 0 \end{array} \right) (B.2)
\times (2K + 1)^{1/2} \left( T^K (\bar{K} k_P) || T^{K'} (\bar{K}' k' P) || T^1 (10) \right).
$$

The reduced supermatrix elements can be expressed as\(^1\)

$$
\left( T^K (\bar{K} k_P) || T^1 (10) || T^{K'} (\bar{K}' k' P) \right) = [3 (2K' + 1) (2\bar{K} + 1) (2k_P + 1)]^{1/2}
\times \left\{ \begin{array}{cc} K & K' & 1 \\ \bar{K} & \bar{K}' & 1 \\ k_P & k'_P & 0 \end{array} \right\}
\left( \begin{array}{cc} T^K (k_I k_S || T^1 (10) || T^{K'} (k'_I k'_S) \right) \left( k_P || T^0 (P) || k' P \right),
$$

(B.5)
and
\[
\langle T^K(\vec{K}k_P) || T^{K'}(\vec{K}'k'_P) || T^1(10) \rangle = [3 (2K' + 1) (2\vec{K} + 1) (2k_P + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
K & 1 & K' \\
\vec{K} & 1 & \vec{K}' \\
k_P & 0 & k'_P
\end{array} \right\} \langle T^K(k_Ik_S) || T^{K'}(k'_Ik'_S) || T^1(10) \rangle \langle k_P || T^{k'_P}(P) || 0 \rangle. \tag{B.6}
\]

Note that the argument (10) in \( T^1(10) \) refers to \((\vec{K}k_P)\) in Eqs. (B.2) – (B.4) and on the left-hand side of Eqs. (B.5) and (B.6), but to \((k_Ik_S)\) on the right-hand side of Eqs. (B.5) and (B.6).

The single-spin reduced supermatrix elements in Eqs. (B.5) and (B.6) can be expressed in terms of a \( 6j \) symbol as
\[
\langle k_P || T^0(P) || k'_P \rangle = \langle k_P || T^{k'_P}(P) || 0 \rangle
\]
\[
= (-1)^{k'_P+1} (2k'_P + 1)^{1/2} \left\{ \begin{array}{ccc}
k_P & k'_P & 0 \\
1/2 & 1/2 & 1/2
\end{array} \right\}. \tag{B.7}
\]

The two-spin reduced supermatrix elements in Eqs. (B.5) and (B.6) can be expressed in terms of single-spin reduced supermatrix elements, as in Eqs. (B.5) and (B.6),
\[
\langle T^K(k_Ik_S) || T^1(10) || T^{K'}(k'_Ik'_S) \rangle = [3 (2\vec{K}' + 1) (2k_I + 1) (2k_S + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
\vec{K} & \vec{K}' & 1 \\
k_I & k'_I & 1 \\
k_S & k'_S & 0
\end{array} \right\} \langle k_I || T^1(I) || k'_I \rangle \langle k_S || T^0(S) || k'_S \rangle, \tag{B.8}
\]
and
\[
\langle T^K(k_Ik_S) || T^{K'}(k'_Ik'_S) || T^1(10) \rangle = [3 (2\vec{K}' + 1) (2k_I + 1) (2k_S + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
\vec{K} & 1 & \vec{K}' \\
k_I & 1 & k'_I \\
k_S & 0 & k'_S
\end{array} \right\} \langle k_I || T^{k'_I}(I) || 1 \rangle \langle k_S || T^{k'_S}(S) || 0 \rangle. \tag{B.9}
\]

The single-spin reduced supermatrix elements in Eqs. (B.8) and (B.9) can be expressed in
terms of $6j$ symbols as

$$\langle k_I \| T^I (I) \| k_I' \rangle \langle k_S \| T^0 (S) \| k_S' \rangle = \langle k_I \| T^{k_I} (I) \| 1 \rangle \langle k_S \| T^{k_S} (S) \| 0 \rangle$$

$$= (-1)^{k_I+k_S+1} \left[ 3 (2k_I+1)(2k_S+1) \right]^{1/2} \left\{ \begin{array}{ccc} k_I & k_I' & 1 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} k_S & k_S' & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} . \quad (B.10)$$

Combining Eqs. (B.2) – (B.10) and using the symmetries of the Wigner $3j$, $6j$ and $9j$ symbols with respect to column permutations, we find

$$\left( T^K_Q (k_I k_S \{ K \} k_P) | I_z | T^{K'}_{Q'} (k_I' k_S' \{ K' \} k_P') \right) = \delta_{QQ'} (-1)^{K'-Q+1} \Delta W$$

$$\times \left\{ \begin{array}{ccc} K & K' & 1 \\ -Q & Q & 0 \end{array} \right\} \left\{ \begin{array}{ccc} k_I & k_I' & 1 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} k_S & k_S' & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} k_P & k_P' & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} . \quad (B.11)$$

where

$$\Delta \equiv (-1)^{k_I+k_S+k_P} + (-1)^{k_I'+k_S'+k_P'} , \quad (B.12)$$

and

$$W \equiv 3 [6 (2K+1) (2K'+1) (2\bar{K}+1) (2\bar{K}'+1)$$

$$\times (2k_I+1) (2k_I'+1) (2k_S+1) (2k_S'+1) (2k_P+1) (2k_P'+1) ]^{1/2} . \quad (B.13)$$

Using the identity, for integral $a$ and $b$,\textsuperscript{1}

$$\left\{ \begin{array}{ccc} a & b & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} = \delta_{ab} \frac{(-1)^{a+1}}{2(2a+1)]^{1/2} ,} \quad (B.14)$$

and noting that $k_I, k_I' = 0$ or 1, we see that

$$\Delta \left\{ \begin{array}{ccc} k_S & k_S' & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} k_P & k_P' & 0 \\ 1/2 & 1/2 & 1/2 \end{array} \right\} = \delta_{k_I k_I'} \delta_{k_S k_S'} \delta_{k_P k_P'} \frac{(1-2k_I)}{(2k_S+1) (2k_P+1)]^{1/2} .} \quad (B.15)$$

Next, we rewrite the $9j$ symbols in Eq. (B.11) with the aid of the identity\textsuperscript{1}

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & c \\ g & h & 0 \end{array} \right\} = \delta_{gb} \frac{(-1)^{b+d+g+c}}{[(2c+1) (2g+1)]^{1/2} \left\{ \begin{array}{ccc} a & b & c \\ e & d & g \end{array} \right\} .} \quad (B.16)$$
Combining Eqs. (B.11), (B.15) and (B.16), and noting that, since \( k_I = 0 \) or 1,

\[
\begin{pmatrix}
  k_I & k_I & 1 \\
  \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix} = \delta_{k_I 1} \left(-\frac{1}{3}\right),
\] (B.17)

we obtain

\[
\left( T_Q^K (k_I k_S \{K\} k_P) | \mathcal{L}_z | T_Q^{K'} (k'_I k'_S \{K'\} k'_P) \right) = \delta_{QQ'} \delta_{kk'} \delta_{k_S k'_S} \delta_{k_P k'_P} \delta_{k_I 1} 
\]
\[
\times (-1)^{K+K'+k_S+k_P-Q} [6 (2K + 1) (2K' + 1) (2\bar{K} + 1) (2\bar{K}' + 1)]^{1/2}
\]
\[
\times \left( \begin{array}{cc}
  K & K' \\
  -Q & Q
\end{array} \right) \left\{ \begin{array}{ccc}
  K & K' & 1 \\
  \bar{K}' & \bar{K} & k_P
\end{array} \right\} \left\{ \begin{array}{c}
  \bar{K} \bar{K}' 1 \\
  1 \ 1 \ k_S
\end{array} \right\}.
\] (B.18)

In the same way, we find

\[
\left( T_Q^K (k_I k_S \{K\} k_P) | \mathcal{S}_z | T_Q^{K'} (k'_I k'_S \{K'\} k'_P) \right) = \delta_{QQ'} \delta_{kk'} \delta_{k_S k'_S} \delta_{k_P k'_P} \delta_{k_S 1}
\]
\[
\times (-1)^{k_I+k_P-Q} [6 (2K + 1) (2K' + 1) (2\bar{K} + 1) (2\bar{K}' + 1)]^{1/2}
\]
\[
\times \left( \begin{array}{cc}
  K & K' \\
  -Q & Q
\end{array} \right) \left\{ \begin{array}{ccc}
  K & K' & 1 \\
  \bar{K}' & \bar{K} & k_P
\end{array} \right\} \left\{ \begin{array}{c}
  \bar{K} \bar{K}' 1 \\
  1 \ 1 \ k_I
\end{array} \right\},
\] (B.19)

and

\[
\left( T_Q^K (k_I k_S \{K\} k_P) | \mathcal{P}_z | T_Q^{K'} (k'_I k'_S \{K'\} k'_P) \right) = \delta_{QQ'} \delta_{kk'} \delta_{k_S k'_S} \delta_{k_P k'_P} \delta_{k_P 1} \delta_{\bar{K}\bar{K}'}
\]
\[
\times (-1)^{K+K'+k_I+k_S+1-Q} [6 (2K + 1) (2K' + 1) (2\bar{K} + 1) (2k_I + 1)]^{1/2}
\]
\[
\times \left( \begin{array}{cc}
  K & K' \\
  -Q & Q
\end{array} \right) \left\{ \begin{array}{ccc}
  K & K' & 1 \\
  1 \ 1 \ \bar{K}
\end{array} \right\} \left\{ \begin{array}{c}
  k_I \ k_I \ k_S
\end{array} \right\}.
\] (B.20)

Using the identity

\[
\left\{ \begin{array}{cccc}
  \bar{K} & \bar{K} & 0 \\
  k_I & k_I & k_S
\end{array} \right\} = \frac{(-1)^{K+K_I+k_S}}{[(2\bar{K} + 1) (2k_I + 1)]^{1/2}},
\] (B.21)

we can also rewrite Eq. (B.20) as

\[
\left( T_Q^K (k_I k_S \{K\} k_P) | \mathcal{P}_z | T_Q^{K'} (k'_I k'_S \{K'\} k'_P) \right) = \delta_{QQ'} \delta_{kk'} \delta_{k_S k'_S} \delta_{k_P k'_P} \delta_{k_P 1} \delta_{\bar{K}\bar{K}'}
\]
\[
\times (-1)^{K+K'+k+1-Q} [6 (2K + 1) (2K' + 1)]^{1/2}
\]
\[
\times \left( \begin{array}{cc}
  K & K' \\
  -Q & Q
\end{array} \right) \left\{ \begin{array}{ccc}
  K & K' & 1 \\
  1 \ 1 \ \bar{K}
\end{array} \right\}.
\] (B.22)
which is independent of $I$ and $S$ except for the Kronecker deltas. Note that the supermatrix elements in Eqs. (B.18) and (B.19) are not simply related by an $I \leftrightarrow S$ interchange, because all basis operators are not invariant under this permutation.

According to Eq. (1), the Zeeman Liouvillian is

$$L_Z = \omega_I (I_z + S_z + P_z) + \omega_I (\delta_S S_z + \delta_P P_z).$$  \hspace{1cm} (B.23)

The supermatrix representation of $I_z + S_z + P_z$ is diagonal with the diagonal elements equal to $Q$, so

$$(n \mid L_Z \mid n') = \delta_{nn'} \omega_I Q + \omega_I [\delta_S (n \mid S_z \mid n') + \delta_P (n \mid P_z \mid n')],$$  \hspace{1cm} (B.24)

with the matrix elements $(n \mid S_z \mid n')$ and $(n \mid P_z \mid n')$ given by Eqs. (B.19) and (B.22), respectively. For isochronous spins, with $\delta_S = \delta_P = 0$ so that $\omega_I = \omega_S = \omega_P \equiv \omega_0$,

$$(n \mid L_Z \mid n') = \delta_{nn'} \omega_0 Q.$$  \hspace{1cm} (B.25)
APPENDIX C: DIPOLAR SUPERMATRIX

Here we calculate the supermatrix representation of the dipolar Liouvillian $L^\alpha_D = [H^\alpha_D, ]$ in the ISTO basis of Table S1. According to Eq. (3), the first part of the dipolar Liouvillian, corresponding to the $I-S$ dipole coupling, is

$$L^\alpha_{D,IS} = -\frac{2}{\sqrt{3}} \omega_{D,IS} \sum_{M=-2}^{2} T_M^2(11\{2\}0) D_{M0}^2(\Omega^\alpha_{IS}), \quad (C.1)$$

where $T_M^2 = |T_M^2,]$. We thus consider the matrix element

$$\langle n | T_M^2(11\{2\}0) | n' \rangle = \left( T_Q^K(k_I k_S \{ \bar{K} \} k_P) | T_M^2(11\{2\}0) | T_{Q'}^{K'}(k'_I k'_S \{ \bar{K}' \} k'_P) \right)$$

$$- \left( T_Q^K(k_I k_S \{ \bar{K} \} k_P) | T_{Q'}^{K'}(k'_I k'_S \{ \bar{K}' \} k'_P) | T_M^2(11\{2\}0) \right). \quad (C.2)$$

Regarding the coupled $I$ and $S$ spins as a composite system, we can write this as

$$\langle n | T_M^2(11\{2\}0) | n' \rangle = \left( T_Q^K(\bar{K} k_P) | T_M^2(20) | T_{Q'}^{K'}(\bar{K}' k'_P) \right)$$

$$- \left( T_Q^K(\bar{K} k_P) | T_{Q'}^{K'}(\bar{K}' k'_P) | T_M^2(20) \right). \quad (C.3)$$

According to the Wigner-Eckart theorem,\(^1\)

$$\left( T_Q^K(\bar{K} k_P) | T_M^2(20) | T_{Q'}^{K'}(\bar{K}' k'_P) \right) = \delta_{M,Q-Q'}(-1)^{K-Q} (2K+1)^{1/2}$$

$$\times \left( \begin{array}{ccc} K & 2 & K' \\ -Q & Q - Q' & Q' \end{array} \right) \left< T^K(\bar{K} k_P) \parallel T^2(20) \parallel T^{K'}(\bar{K}' k'_P) \right>, \quad (C.4)$$

and

$$\left( T_Q^K(\bar{K} k_P) | T_{Q'}^{K'}(\bar{K}' k'_P) | T_M^2(20) \right) = \delta_{M,Q-Q'}(-1)^{K-Q} (2K+1)^{1/2}$$

$$\times \left( \begin{array}{ccc} K & K' & 2 \\ -Q & Q' & Q - Q' \end{array} \right) \left< T^K(\bar{K} k_P) \parallel T^{K'}(\bar{K}' k'_P) \parallel T^2(20) \right>. \quad (C.5)$$

The reduced supermatrix elements can be expressed as\(^1\)

$$\left< T^K(\bar{K} k_P) \parallel T^2(20) \parallel T^{K'}(\bar{K}' k'_P) \right> = [5 (2K' + 1)(2\bar{K} + 1)(2k_P + 1)]^{1/2}$$

$$\times \left\{ \begin{array}{ccc} K & K' & 2 \\ \bar{K} & \bar{K}' & 2 \\ k_P & k'_P & 0 \end{array} \right\} \left< T^K(k_I k_S) \parallel T^2(11) \parallel T^{\bar{K}'}(k'_I k'_S) \right> \left< k_P \parallel T^0(P) \parallel k'_P \right>, \quad (C.6)$$

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and
\[
\left\langle T^K(\bar{K}k_P) \middle| T^{K'}(\bar{K}'k'_P) \middle| T^2(20) \rightangle = [5(2K' + 1)(2\bar{K} + 1)(2k_P + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
K & 2 & K' \\
\bar{K} & 2 & \bar{K}' \\
k_P & 0 & k'_P
\end{array} \right\} \left\langle T^\bar{K}(k_Ik_S) \middle| T^{\bar{K}'}(k'_Ik'_S) \middle| T^2(11) \rightangle \left\langle k_P \middle| T^{k_P}(P) \middle| 0 \right\rangle .
\] (C.7)

The single-spin reduced supermatrix elements in Eqs. (C.6) and (C.7) are given by Eq. (B.7), and the two-spin reduced supermatrix elements can be expressed in terms of single-spin reduced supermatrix elements as
\[
\left\langle T^\bar{K}(k_Ik_S) \middle| T^2(11) \middle| T^{\bar{K}'}(k'_Ik'_S) \rightangle = [5(2\bar{K}' + 1)(2k_I + 1)(2k_S + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
\bar{K} & \bar{K}' & 2 \\
k_I & k'_I & 1 \\
k_S & k'_S & 1
\end{array} \right\} \left\langle k_I \middle| T^1(I) \middle| k'_I \right\rangle \left\langle k_S \middle| T^1(S) \middle| k'_S \right\rangle ,
\] (C.8)

and
\[
\left\langle T^\bar{K}(k_Ik_S) \middle| T^{\bar{K}'}(k'_Ik'_S) \middle| T^2(11) \rightangle = [5(2\bar{K}' + 1)(2k_I + 1)(2k_S + 1)]^{1/2}
\]
\[
\times \left\{ \begin{array}{ccc}
\bar{K} & 2 & \bar{K}' \\
k_I & 1 & k'_I \\
k_S & 1 & k'_S
\end{array} \right\} \left\langle k_I \middle| T^{k'_I}(I) \middle| 1 \right\rangle \left\langle k_S \middle| T^{k'_S}(S) \middle| 1 \right\rangle .
\] (C.9)

The single-spin reduced supermatrix elements in Eqs. (C.8) and (C.9) can be expressed in terms of $6j$ symbols as
\[
\left\langle k_I \middle| T^1(I) \middle| k'_I \right\rangle \left\langle k_S \middle| T^1(S) \middle| k'_S \right\rangle = \left\langle k_I \middle| T^{k'_I}(I) \middle| 1 \right\rangle \left\langle k_S \middle| T^{k'_S}(S) \middle| 1 \right\rangle = (-1)^{k'_I+k'_S} 3[(2k'_I + 1)(2k'_S + 1)]^{1/2}
\]
\[
\left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
1/2 & 1/2 & 1/2
\end{array} \right\} \left\{ \begin{array}{ccc}
k_S & k'_S & 1 \\
1/2 & 1/2 & 1/2
\end{array} \right\} .
\] (C.10)

Combining Eqs. (C.2) – (C.10) and using the symmetries of the Wigner $3j$, $6j$ and $9j$ symbols with respect to column permutations, we find
\[(n \mid T_M^{2}(11\{2\}0) \mid n') = \delta_{M,Q-Q'} (-1)^{K'-Q} \left[ (-1)^{k_I+k_S+k_P} - (-1)^{k'_I+k'_S+k'_P} \right] \]
\[
\times 15 \left[ (2K + 1)(2K' + 1)(2\bar{K} + 1)(2\bar{K}' + 1)(2k_I + 1)(2k'_I + 1)(2k_S + 1)(2k'_S + 1) \right]^{1/2}
\times (2k_P + 1)(2k'_P + 1) \right)^{1/2}
\times \left\{ \begin{array}{ccc}
K & K' & 2 \\
\bar{K} & \bar{K}' & 2 \\
k_P & k'_P & 0 \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right\}
\right].
\]

The last 6j symbol in Eq. (C.11) may be expressed with the aid of Eq. (B.14). For the second 9j symbol, Eq. (B.16) yields

\[
\left\{ \begin{array}{ccc}
K & K' & 2 \\
\bar{K} & \bar{K}' & 2 \\
k_P & k'_P & 0 \\
\end{array} \right\} = \delta_{k_P k'_P} \left( -1 \right)^{K'+\bar{K}+k_P} \frac{1}{5(2k_P + 1)} \left[ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right] \left[ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right] \left[ \begin{array}{ccc}
k_I & k'_I & 1 \\
k_S & k'_S & 1 \\
k_P & k'_P & 0 \\
\end{array} \right].
\]

Combining Eqs. (C.11), (C.12) and (B.14) with Eq. (C.1) and noting that (since \( k_I, k'_I, k_S, k'_S = 0 \) or 1)

\[
\left[ (-1)^{k_I+k_S} - (-1)^{k'_I+k'_S} \right] = 2 \left( \delta_{k_I k_S} - \delta_{k'_I k'_S} \right),
\]

we obtain

\[
(n \mid L_{D,IS}^{a} \mid n') = \left( \delta_{k_I k_S} - \delta_{k'_I k'_S} \right) \delta_{k_P k'_P} \left( -1 \right)^{K'+k_P-Q} 2 \sqrt{30} \omega_{D,IS} D^{2*}_{Q-Q',0}(\Omega_{IS})
\times \left[ (2K + 1)(2K' + 1)(2\bar{K} + 1)(2\bar{K}' + 1)(2k_I + 1)(2k'_I + 1)(2k_S + 1)(2k'_S + 1) \right]^{1/2}
\times \left\{ \begin{array}{ccc}
K & K' & 2 \\
-Q & Q' & Q-Q' \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_S & k'_S & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_I & k'_I & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_S & k'_S & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\} \left\{ \begin{array}{ccc}
k_P & k'_P & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\}
\right].
\]
Since the first factor, \((\delta_{k I k S} - \delta_{k'I k'S})\), vanishes if \(k_I = k'_I\) and \(k_S = k'_S\), it follows that \((n | L_{D,IS}^a | n) = 0\), that is, only off-diagonal \((n \neq n')\) supermatrix elements can be nonzero.

In the same way, we find

\[ (n | L_{D,IP}^a | n') = \left(\delta_{k I k P} - \delta_{k'I k'P}\right) \delta_{k S k' P} (-1)^{K' + K + k I + k S - Q + 1} 2 \sqrt{30} \omega_{D,IP} D_{Q-Q',0}^2 (\Omega_{IP}^a) \]

\[ \times \left[ (2K + 1) (2K' + 1) (2\bar{K} + 1) (2\bar{K}' + 1) (2k_I + 1) (2k'_I + 1) (2k_P + 1) (2k'_P + 1) \right]^{1/2} \]

\[ \times \left( \begin{array}{ccc} K & K' & 2 \\ -Q & Q' & Q - Q' \end{array} \right) \left( \begin{array}{ccc} k_I & k'_I & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} k_P & k'_P & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} \bar{K} & \bar{K}' & 1 \\ k'_I & k_I & k_S \end{array} \right) \left( \begin{array}{ccc} K & K' & 2 \\ \bar{K} & \bar{K}' & 1 \\ k_P & k'_P & 1 \end{array} \right) \] (C.15)

and

\[ (n | L_{D,SP}^a | n') = \left(\delta_{k_s k_P} - \delta_{k'_s k'_P}\right) \delta_{k I k P} (-1)^{K' + K + k S + k P - Q + 1} 2 \sqrt{30} \omega_{D,SP} D_{Q-Q',0}^2 (\Omega_{SP}^a) \]

\[ \times \left[ (2K + 1) (2K' + 1) (2\bar{K} + 1) (2\bar{K}' + 1) (2k_S + 1) (2k'_S + 1) (2k_P + 1) (2k'_P + 1) \right]^{1/2} \]

\[ \times \left( \begin{array}{ccc} K & K' & 2 \\ -Q & Q' & Q - Q' \end{array} \right) \left( \begin{array}{ccc} k_S & k'_S & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} k_P & k'_P & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} \bar{K} & \bar{K}' & 1 \\ k'_S & k_S & k_I \end{array} \right) \left( \begin{array}{ccc} K & K' & 2 \\ \bar{K} & \bar{K}' & 1 \\ k_P & k'_P & 1 \end{array} \right) \] (C.16)
APPENDIX D: INTEGRAL RELAXATION RATE

Here we present explicit expressions, based on Eqs. (19) and (20), for the integral longitudinal relaxation rate in terms of the matrix elements $U_{np}^{XY} \equiv \langle n|\tilde{U}^{XY}(0)|p \rangle$ and the relative magnetogyric ratios $\kappa_S \equiv \gamma_S/\gamma_I$ and $\kappa_P \equiv \gamma_P/\gamma_I$. Results are listed separately for the three exchange cases. Whereas Eqs. (21) – (24) of the main text are restricted to the dilute regime ($P_A \ll 1$), the results given here are valid for arbitrary $P_A$. Excitation is either nonselective (‘non’), meaning that all three spins are excited (as in a field-cycling experiment), or selective (‘sel’), meaning that the labile spin(s) (present in both states) but not the nonlabile spin(s) (present only in state A) are excited.

1. Exchange case $ISP-I$

If only one of the three spins is observed, we have for the $I$-magnetization,

$$\hat{R}_{I,I}^{\text{non}} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BA} + U_{12}^{AA}) + \kappa_P(U_{13}^{BA} + U_{13}^{AA})]^{-1},$$  \hspace{1cm} (D.1a)

$$\hat{R}_{I,I}^{\text{sel}} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA}]^{-1},$$  \hspace{1cm} (D.1b)

for the $S$-magnetization,

$$\hat{R}_{I,S}^{\text{non}} = \kappa_S P_A [U_{21}^{AB} + U_{21}^{AA} + \kappa_S U_{22}^{AA} + \kappa_P U_{23}^{AA}]^{-1},$$  \hspace{1cm} (D.2)

and for the $P$-magnetization,

$$\hat{R}_{I,P}^{\text{non}} = \kappa_P P_A [U_{31}^{AB} + U_{31}^{AA} + \kappa_S U_{32}^{AA} + \kappa_P U_{33}^{AA}]^{-1}.$$  \hspace{1cm} (D.3)

If the combined magnetization of spins $I$ and $S$ is observed,

$$\hat{R}_{I,IS}^{\text{non}} = (1 + \kappa_S P_A) [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{21}^{AA} + U_{11}^{AA} + \kappa_S U_{12}^{BA} + \kappa_S U_{12}^{AA} + \kappa_P U_{13}^{BA} + \kappa_P U_{13}^{AA} + U_{22}^{AA} + U_{23}^{AA}]^{-1},$$  \hspace{1cm} (D.4a)

$$\hat{R}_{I,IS}^{\text{sel}} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{21}^{AA} + U_{11}^{AA} + U_{22}^{AA} + U_{21}^{AA}]^{-1}.$$  \hspace{1cm} (D.4b)

If the combined magnetization of all three spins is observed,

$$\hat{R}_{I,ISP}^{\text{non}} = [1 + (\kappa_S + \kappa_P) P_A] [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{21}^{AA} + U_{11}^{AA} + U_{11}^{AA} + U_{31}^{AA} + \kappa_S U_{12}^{BA} + \kappa_S U_{12}^{AA} + U_{32}^{AA} + \kappa_P U_{13}^{BA} + \kappa_P U_{13}^{AA} + U_{33}^{AA} + U_{33}^{AA}]^{-1},$$  \hspace{1cm} (D.5a)

$$\hat{R}_{I,ISP}^{\text{sel}} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{21}^{AA} + U_{11}^{AA} + U_{21}^{AA} + U_{31}^{AA} + U_{31}^{AA}]^{-1}.$$  \hspace{1cm} (D.5b)
2. Exchange case \(ISP-IS\)

If only one of the three spins is observed, we have for the \(I\)-magnetization,

\[
\hat{R}^{\text{non}}_{1,I} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})
+ \kappa_P(U_{13}^{BA} + U_{13}^{AA})]^{-1},
\]

(D.6a)

\[
\hat{R}^{\text{sel}}_{1,I} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})]^{-1},
\]

(D.6b)

for the \(S\)-magnetization,

\[
\hat{R}^{\text{non}}_{1,S} = \kappa_S[U_{21}^{BB} + U_{21}^{BA} + U_{21}^{AB} + U_{21}^{AA} + \kappa_S(U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA})
+ \kappa_P(U_{23}^{BA} + U_{23}^{AA}]^{-1},
\]

(D.7a)

\[
\hat{R}^{\text{sel}}_{1,S} = \kappa_S[U_{21}^{BB} + U_{21}^{BA} + U_{21}^{AB} + U_{21}^{AA} + \kappa_S(U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA})]^{-1},
\]

(D.7b)

and for the \(P\)-magnetization,

\[
\hat{R}^{\text{non}}_{1,P} = \kappa_P P_A [U_{31}^{AB} + U_{31}^{AA} + \kappa_S(U_{32}^{AB} + U_{32}^{AA}) + \kappa_P U_{33}^{AA}]^{-1}.
\]

(D.8)

If the combined magnetization of spins \(I\) and \(S\) is observed,

\[
\hat{R}^{\text{non}}_{1,IS} = (1 + \kappa_S)[U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})
+ \kappa_P(U_{13}^{BA} + U_{13}^{AA})]^{-1},
\]

(D.9a)

\[
\hat{R}^{\text{sel}}_{1,IS} = (1 + \kappa_S)[U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})
+ \kappa_S(U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA})]^{-1}.
\]

(D.9b)

If the combined magnetization of all three spins is observed,

\[
\hat{R}^{\text{non}}_{1,ISP} = (1 + \kappa_S + \kappa_P P_A)[U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})
+ \kappa_P(U_{13}^{BA} + U_{13}^{AA}) + U_{31}^{AA}]^{-1},
\]

(D.10a)

\[
\hat{R}^{\text{sel}}_{1,ISP} = (1 + \kappa_S)[U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} + \kappa_S(U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA})
+ \kappa_S(U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA})]^{-1}.
\]

(D.10b)
3. Exchange case \( ISP-ISP \)

If only one of the three spins is observed, we have for the \( I \)-magnetization,

\[
\hat{R}_{1,I}^{\text{non}} = \hat{R}_{1,I}^{\text{sel}} = [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} \\
+ \kappa_S (U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA}) + \kappa_P (U_{13}^{BB} + U_{13}^{BA} + U_{13}^{AB} + U_{13}^{AA})]^{-1},
\]

for the \( S \)-magnetization,

\[
\hat{R}_{1,S}^{\text{non}} = \hat{R}_{1,S}^{\text{sel}} = \kappa_S [U_{21}^{BB} + U_{21}^{BA} + U_{21}^{AB} + U_{21}^{AA} \\
+ \kappa_S (U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA}) + \kappa_P (U_{23}^{BB} + U_{23}^{BA} + U_{23}^{AB} + U_{23}^{AA})]^{-1},
\]

and for the \( P \)-magnetization,

\[
\hat{R}_{1,P}^{\text{non}} = \hat{R}_{1,P}^{\text{sel}} = \kappa_P [U_{31}^{BB} + U_{31}^{BA} + U_{31}^{AB} + U_{31}^{AA} \\
+ \kappa_S (U_{32}^{BB} + U_{32}^{BA} + U_{32}^{AB} + U_{32}^{AA}) + \kappa_P (U_{33}^{BB} + U_{33}^{BA} + U_{33}^{AB} + U_{33}^{AA})]^{-1}.
\]

If the combined magnetization of spins \( I \) and \( S \) is observed,

\[
\hat{R}_{1,IS}^{\text{non}} = \hat{R}_{1,IS}^{\text{sel}} = (1 + \kappa_S) [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} \\
+ U_{21}^{AA} + \kappa_S (U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA}) + \kappa_P (U_{13}^{BB} + U_{13}^{BA} + U_{13}^{AB} + U_{13}^{AA})]^{-1}.
\]

If the combined magnetization of all three spins is observed,

\[
\hat{R}_{1,ISP}^{\text{non}} = \hat{R}_{1,ISP}^{\text{sel}} = (1 + \kappa_S + \kappa_P) [U_{11}^{BB} + U_{11}^{BA} + U_{11}^{AB} + U_{11}^{AA} \\
+ U_{21}^{BA} + U_{21}^{AB} + U_{21}^{BB} + U_{21}^{BB} + \kappa_S (U_{12}^{BB} + U_{12}^{BA} + U_{12}^{AB} + U_{12}^{AA}) + \kappa_P (U_{13}^{BB} + U_{13}^{BB} \\
+ U_{22}^{BB} + U_{22}^{BA} + U_{22}^{AB} + U_{22}^{AA} + U_{22}^{AA}) + \kappa_P (U_{13}^{BB} + U_{13}^{BB} \\
+ U_{33}^{BB} + U_{33}^{BA} + U_{33}^{AB} + U_{33}^{BB} + U_{33}^{BA} + U_{33}^{BA} + U_{33}^{AB} + U_{33}^{AA})]^{-1}.
\]

4. Exchange case \( ISP-I \) in the dilute regime

In the dilute regime \( (P_A \ll 1) \), Eqs. (D.1), (D.4) and (D.5) reduce to

\[
\hat{R}_{1,I}^{\text{dil}} = \hat{R}_{1,IS}^{\text{dil}} = \hat{R}_{1,ISP}^{\text{dil}} = \frac{1}{U_{11}^{BB}}.
\]

We shall now show that \( U_{11}^{BB} \) can be expressed in terms of the element \( g_{11} \equiv \langle 1|G^A(0)|1 \rangle \) of the supermatrix \( G^A(0) \) in Eq. (14).
Combining Eqs. (12a) and (13) and the detailed balance relation $P_B \tau_A = P_A \tau_B$, we obtain

$$\tilde{U}_{BB}^{(0)} = \frac{P_B^2 \tau_A}{P_A} \left[ P^B + i L^B \tau_B - T G^A(0) T' \right]^{-1}. \quad (D.17)$$

Because the matrix $G^A(0)$ is isotropically averaged, it must reflect the axial symmetry in spin Liouville space. According to the Wigner-Eckart theorem, $G^A(0)$ must then be block-diagonal in the projection index $Q$ of the ISTO basis (Table S1). Pre- and post-multiplication by the $ISP-I$ matrix $T$ in Eq. (15) and its transpose picks out the first diagonal element in each of the first three blocks ($Q = 0, \pm 1$), that is,

$$T G^A(0) T' = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{20,20} & 0 \\ 0 & 0 & g_{35,35} \end{bmatrix}. \quad (D.18)$$

Only spin $I$ can access state B in the $ISP-I$ case, so the Zeeman Liouvillian supermatrix in the B-state basis is simply

$$L^B_Z = \omega^B_I \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (D.19)$$

Combination of Eqs. (D.17) – (D.19) yields

$$\tilde{U}_{BB}^{(0)} = \frac{P_B^2 \tau_A}{P_A} \begin{bmatrix} (1 - g_{11})^{-1} & 0 & 0 \\ 0 & (1 - g_{20,20} + i \omega_I^B \tau_B)^{-1} & 0 \\ 0 & 0 & (1 - g_{35,35} - i \omega_I^B \tau_B)^{-1} \end{bmatrix}, \quad (D.20)$$

so that

$$U_{11}^{BB} = \frac{P_B^2 \tau_A}{P_A(1 - g_{11})}. \quad (D.21)$$

Finally, combination of Eqs. (D.16) and (D.21) yields for the dilute regime

$$\hat{R}_{1,I}^{dil} = \hat{R}_{1,IS}^{dil} = \hat{R}_{1,ISP}^{dil} = \frac{P_A}{\tau_A} (1 - g_{11}). \quad (D.22)$$
Here we obtain the elements of the orientation-dependent relaxation supermatrix $R^\alpha$ in site $\alpha$, starting from the Bloch-Wangsness-Redfield (BWR) master equation

$$\frac{d}{dt} \hat{\sigma}^\alpha(t) = -i \hat{\Delta}^\alpha(t) \hat{\sigma}^\alpha(t) - \int_0^\infty d\tau \left( \hat{L}^\alpha_D(t) \hat{L}^\alpha_D(t - \tau) \right) \hat{\sigma}^\alpha(t), \tag{E.1}$$

where the angular brackets with subscript $\alpha$ denote an equilibrium ensemble average over the molecular degrees of freedom in site $\alpha$. Further, $\hat{\sigma}^\alpha(t) = \exp(i \mathcal{L}_Z t) \sigma^\alpha(t)$ and $\hat{L}^\alpha_D(t) = \exp(i \mathcal{L}_Z t) \mathcal{L}^\alpha_D(t) \exp(-i \mathcal{L}_Z t)$ are the spin density operator and the dipolar Liouvillian for site $\alpha$, both in the interaction representation. For exchange cases $ISP-IS$ and $ISP-ISP$, all three dipole couplings fluctuate so $\mathcal{L}^\alpha_D(t) = \mathcal{L}^\alpha_{D,IS}(t) + \mathcal{L}^\alpha_{D,IP}(t) + \mathcal{L}^\alpha_{D,SP}(t)$ and the first term in Eq. (E.1) is absent. For exchange case $ISP-I$, the $SP$ dipole coupling is static so $\mathcal{L}^\alpha_D(t) = \mathcal{L}^\alpha_{D,IS}(t) + \mathcal{L}^\alpha_{D,IP}(t)$ and, in the first term, $\hat{\Delta}^\alpha(t) \equiv \hat{L}^\alpha_{D,SP}(t) = \exp(i \mathcal{L}_Z t) \mathcal{L}^\alpha_{D,SP} \exp(-i \mathcal{L}_Z t)$.

In a rigorous BWR treatment of the $ISP-I$ case, the transformation to the interaction representation should involve the total static Liouvillian $\mathcal{L}_Z + \mathcal{L}^\alpha_{D,SP}$. The first term in Eq. (E.1) would then be absent, but the ensuing analytical development would be complicated by the fact that the ISTOs $T^K_Q$ are only eigenoperators of $\mathcal{L}_Z$ (see below), not of the total static Liouvillian. The rigorous BWR treatment, which is valid without restrictions on the static dipole coupling $\omega_{D,SP}$, leads to spectral densities at frequencies that are linear combinations of $\omega_0$ and $\omega_{D,SP}$. To avoid these complications, we pursue a restricted BWR treatment, valid for $\omega_{D,SP} \tau_A \ll 1$, where we transform to an intermediate interaction representation as in Eq. (E.1). This additional restriction is unimportant, since, in practice, $\omega_{D,SP}$ cannot exceed $3 \times 10^5$ rad s$^{-1}$, corresponding to the smallest physically realizable proton-proton separation of $\sim 1.5$ Å. If $\tau_A$ is so long that the condition $\omega_{D,SP} \tau_A \ll 1$ is violated, then $\omega_{D,IS}$ and $\omega_{D,IP}$ must be $\ll 10^5$ rad s$^{-1}$ in order not to violate the analogous (motional narrowing) conditions on these fluctuating dipole couplings. But for the dilute conditions ($P_A \ll 1$) of primary interest, the ILRR is negligibly small if all dipole couplings involving the labile $I$ spin are $\ll 10^5$ rad s$^{-1}$. In conclusion, Eq. (E.1) and the results that follow from it, is valid for all three exchange cases, provided that $\omega_{D,X} \tau_A \ll 1$ for all three dipole couplings, regardless of whether they are static or fluctuating.
For simplicity, we assume that the spins are isochronous, that is, \( \delta_A^S = \delta_P^S = 0 \) in Eq. (1) so \( \omega_I^A = \omega_S^A = \omega_P^A \equiv \omega_0 \). The ISTOs in Table S1 are then eigenoperators of the Zeeman Liouvillian,

\[
L_Z T^K_Q = Q \omega_0 T^K_Q ,
\]

so the interaction representation of the Liouvillian for dipole coupling \( X \) in site \( \alpha \) becomes

\[
\hat{L}_{D,X}^\alpha(t) = -\frac{2}{\sqrt{3}} \omega_{D,X} \sum_{M=-2}^{2} \exp(iM\omega_0t) T^2_M(X) D^2_{M0}(\Omega_X^\alpha) ,
\]

where \( T^2_M(X) \equiv [T^2_M(X), \ldots] \). Combination of Eqs. (E.1) and (E.3) yields for the EMOR model

\[
\frac{d}{dt} \hat{\sigma}^\alpha(t) = i \frac{2}{\sqrt{3}} \omega_{D,SP} \sum_{M=-2}^{2} \exp(iM\omega_0t) D^2_{M0}(\Omega_{SP}^\alpha) T^2_M(SP) \hat{\sigma}^\alpha(t)
\]

\[
- \frac{4}{3} \sum_X \sum_Y \omega_{D,X} \omega_{D,Y} \sum_{M=-2}^{2} \sum_{M'=-2}^{2} \exp[i(M+M')\omega_0t]
\]

\[
\times F_{M,M'}(\Omega_X^\alpha, \Omega_Y^\alpha) J(-M'\omega_0) T^2_M(X) T^2_M(Y) \hat{\sigma}^\alpha(t) .
\]

Here, we have introduced the angular functions

\[
F_{M,M'}(\Omega_X^\alpha, \Omega_Y^\alpha) \equiv D^2_{M0}(\Omega_X^\alpha) D^2_{M'0}(\Omega_Y^\alpha) ,
\]

with the Wigner functions for the three internuclear vectors given by Eq. (4), and the reduced spectral density function (SDF)

\[
J(\omega) \equiv j(\omega) + i k(\omega) = \frac{\tau_A}{1 + (\omega \tau_A)^2} (1 + i \omega \tau_A) .
\]

Note that we retain the imaginary part, \( k(\omega) \), of the SDF, which can have a small effect on the ILRR for exchange cases ISP – ISP and ISP – I, but not for the ISP – IS case (Sec. IV C). Because \( k(-\omega) = -k(\omega) \), we refer to \( k(\omega) \) as the odd SDF (OSDF).

Since we want to describe relaxation over the full frequency range, we shall not invoke the secular approximation to eliminate terms with oscillating factors in Eq. (E.4). Instead, we remove these factors by transforming the master equation (E.4) back to the Schrödinger representation. The supermatrix representation of the master equation in the space of the ISTO basis operators \( B_n \) of Table S1 then takes the form

\[
\frac{d}{dt} \sigma^\alpha(t) = - [ i L^A_Z + i \Delta^\alpha + R^\alpha ] \sigma^\alpha(t) ,
\]

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where $\sigma^\alpha(t)$ is a column vector of spin modes $\sigma^\alpha_n(t) = (B_n | \sigma^\alpha(t)) = \text{Tr}\{B^\dagger_n \sigma^\alpha(t)\}$. In view of Eq. (E.4), the static dipolar Liouvillian supermatrix is

$$\Delta^\alpha = -\frac{2}{\sqrt{3}} \omega_{D,SP} \sum_{M=-2}^{2} D^{2*}_{M0}(\Omega^\alpha_{SP}) D_M,$$

(E.8)

and the relaxation supermatrix is

$$R^\alpha = \frac{4}{3} \sum_X \sum_Y \omega_{D,X} \omega_{D,Y} \sum_{M=-2}^{2} \sum_{M'=-2}^{2} F_{MM'}(\Omega^\alpha_X, \Omega^\alpha_Y) J(-M'\omega_0) C^{XY}_{MM'},$$

(E.9)

Note that, in the sums over dipole couplings, $X$ and $Y$ can be IS, IP or SP for exchange cases ISP–IS and ISP–ISP but only IS or IP for exchange case ISP–I, where the static $SP$ coupling appears in $\Delta^\alpha$. In Eqs. (E.8) and (E.9), we have defined the coefficient supermatrices $D_M$ and $C^{XY}_{MM'}$ with elements

$$D_{M,np} \equiv (n | T^2_M(\text{SP}) | p) = \text{Tr}\{B^\dagger_n [T^2_M(\text{SP}), B_p]\},$$

(E.10)

and

$$C^{XY}_{MM',np} \equiv (n | T^2_M(X) T^2_{M'}(Y) | p) = \text{Tr}\{[B^\dagger_n, T^2_M(X)] [T^2_{M'}(Y), B_p]\}.$$  

(E.11)

According to Eq. (E.9), the relaxation supermatrix is determined by $3 \times 3 \times 5 \times 5 = 225$ coefficient supermatrices $C^{XY}_{MM'}$, each with $63 \times 63$ elements. Fortunately, the computational task can be simplified considerably by making use of symmetry. First, we consider spin inversion conjugation (SIC) symmetry.\textsuperscript{5–7} As indicated in Table S1, the ISTO basis operators have definite SIC parity (either odd or even). Furthermore, the relaxation superoperator $R^\alpha$ has even SIC parity.\textsuperscript{5–7} According to the basic orthogonality theorem of group theory,\textsuperscript{8} the supermatrix $R^\alpha$ in the ISTO basis can then have nonzero elements only between basis operators of the same parity. If we order the 63 basis operators so that the 36 odd operators (above the dashed lines in Table S1) precede the 27 even operators, then $R^\alpha$ is block-diagonal,

$$R^\alpha = \begin{bmatrix} R^{\alpha}_{aa} & 0 \\ 0 & R^{\alpha}_{ss} \end{bmatrix},$$

(E.12)

where the subscripts indicate odd (or antisymmetric = a) and even (or symmetric = s) SIC parity. Conversely, the superoperator $\Delta^\alpha$ has odd SIC parity and therefore has nonzero matrix elements between ISTO basis operators of different SIC parity. Consequently, the
supermatrix $\Delta^\alpha$ is anti-block-diagonal in the SIC-parity ordered ISTO basis,

$$\Delta^\alpha = \begin{bmatrix} 0 & \Delta^\alpha_{ss} \\ \Delta^\alpha_{sa} & 0 \end{bmatrix}. \tag{E.13}$$

If we are interested in relaxation of the longitudinal magnetization, described by the odd-parity operators $I_z$, $S_z$, and $P_z$, it might seem that we only need to consider the odd relaxation matrix $R^\alpha_{aa}$. This is true for exchange cases $ISP^{-I}S$ and $ISP^{-ISP}$. However, exchange case $ISP^{-I}$ also involves the static dipolar precession supermatrix $\Delta^\alpha$, which couples the odd and even blocks, $R^\alpha_{aa}$ and $R^\alpha_{ss}$.

Several symmetry relations for the elements of the coefficient matrices $C^{XY}_{M'M; a\alpha b\beta}$ can be derived from the defining relation (E.11). First, it can be shown that all elements of these matrices are real-valued. Using this fact, the cyclic permutation invariance of the trace, and the identities $T_M^2\dagger = (-1)^M T_M^2$ and $(AB)^\dagger = B^\dagger A^\dagger$, we find from Eq. (E.11) that

$$C^{XY}_{M'M; a\alpha b\beta} = (-1)^{M+M'} C^{XY}_{M'-M; a\alpha b\beta}. \tag{E.14}$$

Let $a_\alpha$ and $b_\beta$ denote two of the nine single-spin operators with $a, b = I, S$ or $P$ and $\alpha, \beta = z, +$ or $-$. Within this subspace,

$$C^{XY}_{M'M; a\alpha b\beta} = \delta_{XY} C^{XX}_{M'M; a\alpha b\beta}. \tag{E.15}$$

As a consequence of this selection rule, relaxation matrix elements $R^\alpha_{aa\beta}$ within the single-spin subspace only involve self-correlations ($X = Y$), that is, there is no contribution from distinct correlations ($X \neq Y$). To establish this selection rule, we start from the defining Eq. (E.11): $C^{XY}_{M'M; a\alpha b\beta} = \text{Tr}\{[a_\alpha^\dagger, T_M^2(X)] [T_{M'}^2(Y), b_\beta]\}$. For both commutators to be nonzero, it is necessary that $a \in X$ and $b \in Y$. For example, if $X = IP$, then $a_\alpha$ must be an $I$-spin operator or a $P$-spin operator. The operators $T_M^2(X)$ are products of two single-spin operators (or sums of such products), one for each of the two spins associated with dipole coupling $X$. Therefore, each of the commutators is a product of two single-spin operators (or a sum of such products) associated with the two spins contained in $X$ or $Y$, respectively. It then follows that, if $X \neq Y$, only two of the four operators in the trace belong to the same spin. In other words, two of the three spins are represented by only one operator in the trace. Since $\text{Tr}_a\{a_\alpha\} = 0$, it follows that $C^{XY}_{M'M; a\alpha b\beta} = 0$ if $X \neq Y$. On the other hand, if $X = Y$,
the trace contains two operators associated with each of the two spins contained in X and it can therefore be nonzero.

In a similar way, one can show that $C_{MM',a_α n}^{X X} = 0$ if $a_α$ is a single-spin operator and $n$ is a three-spin basis operator. In other words, the single-spin and three-spin subspaces are coupled in the relaxation matrix only via distinct correlations ($X \neq Y$).

The relaxation supermatrix $R_{α}$ depends on the site orientation $Ω_α$ via the angular functions $F_{MM'}(Ω_α^X, Ω_α^Y)$. Recalling the orthogonality of the Wigner functions, we obtain after averaging these functions over the isotropic distribution of site orientations,

$$\langle F_{MM'} \rangle = \sum_{N=-2}^{2} \langle D_{M0}^2(Ω_α^X) D_{M'N}^2(Ω_α^X) \rangle D_{N0}^{2*}(Ω_{XY})$$

$$= (-1)^M \sum_{N=-2}^{2} \delta_{M',-M} \delta_{N0} \frac{1}{5} D_{N0}^{2*}(Ω_{XY})$$

$$= \delta_{M',-M} (-1)^M \frac{1}{5} P_2(\cos β_{XY}) ,$$

where $β_{XY}$ is the fixed angle between internuclear vectors X and Y (in all sites $α$). Combination of Eqs. (E.9) and (E.16) yields for the isotropically averaged relaxation supermatrix,

$$\langle R_{α} \rangle = \frac{4}{15} \sum_{X} \sum_{Y} ω_{D,X} ω_{D,Y} P_2(\cos β_{XY}) \sum_{M=-2}^{2} (-1)^M J(M, ω_0) C_{XY}^{M',M} .$$

When expressed in the ISTO basis of Table S1, the isotropically averaged relaxation supermatrix $\langle R_{α} \rangle$ is block-diagonal in the projection index $Q$, in accordance with the Wigner-Eckart theorem. Since $R_{α}$ is also block-diagonal with respect to SIC parity, as expressed by Eq. (E.12), it follows that the evolution of the longitudinal magnetization modes under the site-averaged relaxation supermatrix $\langle R_{α} \rangle$ can be fully described within the subspace of the ten odd-parity zero-quantum basis operators above the dashed line in Table S1A. We have previously presented in explicit form the 18 unique coefficient matrices $C_{MM',a_α n}^{X Y}$, then defined as $C_{M}^{XY} \equiv (-1)^M C_{M,-M}^{XY}$.
The secular approximation is valid in the frequency range $\omega_0 \gg \omega_D^2 \tau_A$, where the rapidly oscillating complex exponentials in Eq. (E.4) effectively eliminate all terms except those with $M' = -M$. Consequently,

$$R_{\text{sec}}^\alpha = \frac{4}{3} \sum_X \sum_Y \omega_D X \omega_D Y \sum_{M=-2}^{2} F_{M,-M}(\Omega_X,\Omega_Y) J(M\omega_0) C_{M,-M}^{XY}.$$  \hfill (E.18)

Comparison of Eqs. (E.17) and (E.18) shows that, with regard to rotational symmetry, secular truncation has the same effect as isotropic averaging: all the supermatrices $C_{M,-M}^{XY}$, as well as $R_{\text{sec}}^\alpha$, are block-diagonal in $Q$. Furthermore, within the single-spin subspace, Eq. (E.15) yields

$$C_{M,-M,a\beta}^{XY} = \delta_{XY} C_{M,-M,a\alpha}^{XX} = \delta_{XY} \delta_{\alpha\beta} C_{M,-M,a\alpha}^{XX}.$$  \hfill (E.19)

As a consequence of this selection rule, the secular contributions to relaxation matrix elements in the single-spin subspace only involve auto-mode relaxation, that is, no cross-mode relaxation. To establish the second equality in Eq. (E.19), we start from the definition (E.11): $C_{M,-M,a\beta}^{XY} = \text{Tr}\{[a^\dagger_{\alpha}, T^2_M(X)] [T^2_M(X), b]\}$. For both commutators to be nonzero, it is necessary that $a, b \in X$. The trace then contains a product of two single-spin operators for each of the two spins associated with the dipole coupling $X$. Since the only non-vanishing traces of a product of two single-spin operators are $\text{Tr}\{a_z^2\} = \frac{1}{2}$ and $\text{Tr}\{a_\pm a_\mp\} = 1$, it follows that the total trace vanishes unless $\alpha = \beta$.

The elements of the relaxation supermatrix $R^\alpha$ needed to calculate the ILRR can be obtained from Eqs. (E.9) and (E.11) or, in the case of isotropically averaged rates, from Eq. (E.17). In the following, we present explicit expressions for some of these rates. The isotropically averaged rates required in Eq. (31) for the $ISP-ISP$ case can be obtained from Eq. (E.17) and the coefficient matrices $C_{M,-M}^{XY}$, which have been presented in explicit form.\(^5\) In Eq. (37) for the $ISP-IS$ case, we need the isotropically averaged longitudinal auto-spin and cross-spin rates

$$\langle R_{zz}^{II} \rangle = \frac{2}{45} (\omega_{D,IS}^2 + \omega_{D,IP}^2) [j(0) + 3 j(\omega_0) + 6 j(2\omega_0)],$$  \hfill (E.20a)

$$\langle R_{zz}^{SS} \rangle = \frac{2}{45} (\omega_{D,IS}^2 + \omega_{D,SP}^2) [j(0) + 3 j(\omega_0) + 6 j(2\omega_0)],$$  \hfill (E.20b)

$$\langle R_{zz}^{IS} \rangle = \frac{2}{45} \omega_{D,IS}^2 [- j(0) + 6 j(2\omega_0)].$$  \hfill (E.20c)

In Eq. (41) for the $ISP-I$ case, we only need $\langle R_{zz}^{II} \rangle$, which is still given by Eq. (E20a).
The various local relaxation rates $R_{np}^{\alpha}$ appearing in Eqs. (38) and (42) are all in the single-spin subspace. The local auto-spin rates required for the $ISP - I$ case are

$$R_{SS}^{\alpha} = \rho^{IS}, \quad (E.21a)$$
$$R_{PP}^{\alpha} = \rho^{IP}, \quad (E.21b)$$

and for the $ISP - IS$ case

$$R_{PP}^{\alpha} = \rho^{IP} + \rho^{SP}. \quad (E.22)$$

Here we have introduced the generic auto-spin relaxation matrix

$$\rho^X = \begin{bmatrix}
\rho^X_{zz} & \rho^X_{z+} & \rho^X_{z-} \\
\rho^X_{z+} & \rho^X_{++} & \rho^X_{+-} \\
\rho^X_{z-} & \rho^X_{+} & \rho^X_{--}
\end{bmatrix}, \quad (E.23)$$

with the five unique rates

$$\rho^X_{zz} = \frac{4}{3} \omega^2_{D,X} \left\{ \frac{1}{6} F_{00}(X) j(0) - \frac{1}{2} F_{1-1}(X) j(\omega_0) + F_{2-2}(X) j(2\omega_0) \right\}, \quad (E.24a)$$
$$\rho^X_{++} = (\rho^X_{--})^* = \frac{4}{3} \omega^2_{D,X} \left\{ \frac{5}{12} F_{00}(X) j(0) \\
- \frac{1}{4} F_{1-1}(X) \left[ 3 j(\omega_0) + i k(\omega_0) \right] + \frac{1}{2} F_{2-2}(X) \left[ j(2\omega_0) + i k(2\omega_0) \right] \right\}, \quad (E.24b)$$
$$\rho^X_{zz} = - (\rho^X_{zz})^* = \frac{4}{3} \omega^2_{D,X} \left\{ \frac{1}{4\sqrt{3}} F_{0-1}(X) \left[ 2 j(0) - j(\omega_0) - i k(\omega_0) \right] \\
- \frac{1}{\sqrt{8}} F_{-21}(X) \left[ j(\omega_0) - i k(\omega_0) \right] \right\}, \quad (E.24c)$$
$$\rho^X_{z+} = - (\rho^X_{z+})^* = \frac{4}{3} \omega^2_{D,X} \left\{ \frac{1}{4\sqrt{3}} F_{01}(X) \left[ j(0) - 2 j(\omega_0) + i 2 k(\omega_0) \right] \\
+ \frac{1}{\sqrt{8}} F_{-12}(X) \left[ j(2\omega_0) - i k(2\omega_0) \right] \right\}, \quad (E.24d)$$
$$\rho^X_{z-} = (\rho^X_{z-})^* = \frac{2}{3} \omega^2_{D,X} \left\{ - \frac{1}{\sqrt{6}} F_{02}(X) \left[ j(0) + j(2\omega_0) - i k(2\omega_0) \right] \\
+ \frac{1}{2} F_{11}(X) \left[ j(\omega_0) - i k(\omega_0) \right] \right\}, \quad (E.24e)$$

where $F_{MM'}(X)$ is a short-hand notation for $F_{MM'}(\Omega^\alpha_X, \Omega^\alpha_X)$. 

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The required local cross-spin rates are contained in the row matrices $R_{\alpha IS}$ and $R_{\alpha IP}$ for the $ISP-I$ case, and in $R_{\alpha IP}$ and $R_{\alpha SP}$ for the $ISP-IS$ case. The three elements of these matrices are of the form,

$$R^{X}_{zz} = \frac{4}{3} \omega_{D,X}^{2} \left[ -\frac{1}{6} F_{00}(X) j(0) + F_{2-2}(X) j(2 \omega_{0}) \right] ,$$

(E.25a)

$$R^{X}_{z+} = - (R^{X}_{z-})^{*} = \frac{4}{3} \omega_{D,X}^{2} \left\{ \frac{1}{4\sqrt{3}} F_{0-1}(X) \left[ j(0) + j(\omega_{0}) + ik(\omega_{0}) \right] - \frac{1}{\sqrt{8}} F_{1-2}(X) \left[ j(\omega_{0}) + j(2 \omega_{0}) - ik(\omega_{0}) + ik(2 \omega_{0}) \right] \right\} .$$

(E.25b)

For isochronous spins, the cross-spin rates have the symmetry $R_{\alpha SI} = R_{\alpha IS}$ and similarly for the other two pairs. As can be seen, all the cross-mode rates involve both the even and odd parts of the SDF.

All of the local relaxation rates in Eqs. (E.20) - (E.25) connect single-spin modes and are therefore entirely produced by self-correlations. These rates fully determine the ILRR in the $ISP-IS$ case. For the other two exchange cases, the ILRR is also affected by distinct correlations via those elements in the $10 \times 10$ odd-parity $Q = 0$ block of $\langle R^{\alpha} \rangle$ that involve at least one three-spin zero-quantum coherence ($ISP-IS$ case) or via the elements of the $9 \times 9$ relaxation matrix $R_{SS00}$ spanned by two-spin-$SP$ operators ($ISP-I$ case). These rates can be calculated from Eqs. (E.17) and (E.9), respectively, but they are too numerous to list here.
The ILRR in the ISP−I case also involves the static dipolar Liouvillian supermatrix $\Delta^\alpha$ in Eq. (E.8) or, more precisely, the $3 \times 9$ submatrices connecting single-spin $S$ or $P$ modes with two-spin-$SP$ modes. For example,

$$\Delta^\alpha_{S,s0} = \frac{\omega_{D,SP}}{\sqrt{6}}$$

$$\begin{bmatrix}
0 & 0 & \frac{2}{\sqrt{3}}D_0 & D_1 & D_1 & D_{-1} & D_{-1} & 2D_2 & -2D_{-2} \\
0 & -\sqrt{3}D_{-1} & -D_{-1} & -\frac{1}{\sqrt{3}}D_0 & -\sqrt{3}D_0 & -\sqrt{2}D_{-2} & -\sqrt{2}D_2 & -\sqrt{2}D_1 & 0 \\
0 & \sqrt{3}D_1 & -D_1 & -\sqrt{2}D_2 & \sqrt{2}D_2 & -\frac{1}{\sqrt{3}}D_0 & \sqrt{3}D_0 & 0 & \sqrt{2}D_{-1}
\end{bmatrix},
$$

(E.26)

where $D_M \equiv D_M^2(\Omega^\alpha_{SP})$, the complex conjugate of which is given by Eq. (4c). In Eq. (E.26), the basis ordering for the single-spin-$S$ subspace is $\{S_z/\sqrt{2}, -S_-/2, S_-/2\}$, while the two-spin-$SP$ operators are in the order in which they appear in Table S1, that is, $\{13, 16, 19, 31, 34, 46, 49, 55, 61\}$. The matrix $\Delta^\alpha_{P,s0}$ differs from $\Delta^\alpha_{S,s0}$ only in that the sign is reversed for all elements in columns 3, 4 and 6 (corresponding to the odd-rank basis operators 19, 31 and 46). Further, since the supermatrix $\Delta^\alpha$ is Hermitian, it follows that $\Delta^\alpha_{S,s0} = \Delta^\alpha_{S,s0}$. 

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APPENDIX F: INTEGRAL RELAXATION RATE IN THE MN REGIME

Here we present derivations of explicit expressions for the ILRR in the motional narrowing (MN) regime for exchange cases $ISP-IS$ and $ISP-I$. It is convenient to express $G^A(0)$ from Eq. (26) as

$$ G^A(0) = \langle (\Lambda^\alpha)^{-1} \rangle , \quad (F.1) $$

with

$$ \Lambda^\alpha \equiv K + R^\alpha \tau_A + iL_Z^A \tau_A + i\Delta^\alpha \tau_A, \quad (F.2) $$

where the last term appears only for exchange case $ISP-I$. We shall now use symmetry arguments and the MN condition $\omega_D \tau_A \ll 1$ to simplify the supermatrix $\Lambda^\alpha$. Throughout Appendix F, we assume isochronous spins ($\delta_S^A = \delta_P^A = 0$), with $\omega_I^A = \omega_S^A = \omega_P^A \equiv \omega_0$. In Appendix G, we generalize the treatment to include the effects of chemical shifts.

1. Exchange case $ISP-IS$

To obtain the four elements of $G^A(0)$ that appear in Eq. (35), we need only consider the $36 \times 36$ odd-parity block (Appendix E)

$$ \Lambda_{aa}^\alpha = K_{aa} + R_{aa}^\alpha \tau_A + iL_{Z,aa}^A \tau_A. \quad (F.3) $$

Before invoking the MN approximation, we reorder the 36 odd-parity basis operators so that the three operators (numbered 3, 22 and 37 in Table S1) that only involve spin $P$ are at the end. With $K_{aa}$ from Eq. (18), we can then partition the supermatrices in Eq. (F.3) as

$$ \Lambda_{aa}^\alpha = \begin{bmatrix} (1 + R_{11}^\alpha \tau_A + iL_{Z,11}^A \tau_A) & R_{10}^\alpha \tau_A \\ R_{01}^\alpha \tau_A & (R_{00}^\alpha \tau_A + iL_{Z,00}^A \tau_A) \end{bmatrix}, \quad (F.4) $$

where the subscripts indicate whether $K_{nn} = 1$ or 0. We need only consider the ‘11’ block of the inverse $(\Lambda_{aa}^\alpha)^{-1}$, which is

$$ [(\Lambda_{aa}^\alpha)^{-1}]_{11} = \begin{bmatrix} 1 + R_{11}^\alpha \tau_A + iL_{Z,11}^A \tau_A - R_{10}^\alpha (R_{00}^\alpha + iL_{Z,00}^A)^{-1} R_{01}^\alpha \tau_A \\ R_{01}^\alpha \tau_A \end{bmatrix}^{-1} = Z^{-1} - Z^{-1} \left[ R_{11}^\alpha - R_{10}^\alpha (R_{00}^\alpha + iL_{Z,00}^A)^{-1} R_{01}^\alpha \right] \tau_A Z^{-1}, \quad (F.5) $$
where, in the last step, we have expanded to second order in $\omega_0 \tau_A$ (the MN approximation) and defined $\mathbf{Z} \equiv 1 + i \mathbf{L}^A_{Z,11} \tau_A$. According to Eq. (F.1), isotropic averaging of this result yields the corresponding $33 \times 33$ block of $\mathbf{G}^A(0)$. For the ILRR, we only need elements from the upper left $2 \times 2$ block of $\mathbf{G}^A(0)$, corresponding to the basis operators $I_z/\sqrt{2}$ and $S_z/\sqrt{2}$. In this subspace $\mathbf{L}^A_Z = 0$, so Eq. (F.5) yields

$$
\mathbf{G}^A(0) = 1 - \tilde{\mathbf{R}} \tau_A ,
$$

with

$$
\tilde{\mathbf{R}} \equiv \langle \mathbf{R}^\alpha \rangle - \langle \mathbf{T}^\alpha \rangle .
$$

In the interest of notational economy, we do not explicitly indicate that all quantities in Eqs. (F.6) and (F.7) are $2 \times 2$ matrices in the subspace spanned by the basis operators $I_z/\sqrt{2}$ and $S_z/\sqrt{2}$. Combination of Eqs. (35) and (F.6) yields for the ILRR

$$
\tilde{R}^{\text{dil}}_{1,IS} = P_A \frac{2(\tilde{R}_{II} \tilde{R}_{SS} - \tilde{R}_{IS} \tilde{R}_{SI})}{(\tilde{R}_{II} + \tilde{R}_{SS} - \tilde{R}_{IS} - \tilde{R}_{SI})} ,
$$

where the supermatrix $\tilde{\mathbf{R}}$ is indexed by $I$ and $S$ rather than by the labels $n = 1$ and 2. The first part of $\tilde{\mathbf{R}}$ is the isotropic orientational average of the longitudinal relaxation matrix

$$
\mathbf{R}^\alpha = \begin{bmatrix}
R_{zz}^{II} & R_{zz}^{IS} \\
R_{zz}^{IS} & R_{zz}^{SS}
\end{bmatrix} .
$$

The second part is the isotropic orientational average of the “cross relaxation” supermatrix

$$
\mathbf{\Gamma}^\alpha \equiv \mathbf{R}_{IP/SP}^{\alpha} (\mathbf{R}_{PP}^{\alpha} + i \omega_0 \mathbf{Q})^{-1} \mathbf{R}_{IP/SP}^{\alpha\dagger} ,
$$

where $\mathbf{Q} = \text{diag}(0, 1, -1)$. Further, $\mathbf{R}_{IP/SP}^{\alpha\dagger}$ is the Hermitian conjugate (or conjugate transpose) of the rectangular relaxation matrix

$$
\mathbf{R}_{IP/SP}^{\alpha} = \begin{bmatrix}
\mathbf{R}_{IP}^\alpha \\
\mathbf{R}_{SP}^\alpha
\end{bmatrix} = \begin{bmatrix}
R_{zz}^{IP} & R_{z+}^{IP} & R_{z-}^{IP} \\
R_{zz}^{SP} & R_{z+}^{SP} & R_{z-}^{SP}
\end{bmatrix} ,
$$

and

$$
\mathbf{R}_{PP}^{\alpha} = \begin{bmatrix}
R_{zz}^{PP} & R_{z+}^{PP} & R_{z-}^{PP} \\
R_{zz}^{PP} & R_{z+}^{PP} & R_{z-}^{PP} \\
R_{zz}^{PP} & R_{z+}^{PP} & R_{z-}^{PP}
\end{bmatrix} .
$$
In Eqs. (F.9) and (F.10), we have made use of the following symmetry relations: \( R_{zz}^{SI} = R_{zz}^{IS} \), \( R_{PI}^{\alpha} = R_{IP}^{\alpha} \), \( R_{PS}^{\alpha} = R_{SP}^{\alpha} \), \( R_{zz}^{IP} = R_{zz}^{IP}^{*} \) and \( R_{ZZ}^{SP} = R_{ZZ}^{SP}^{*} \). All these relations, except the first one, are only valid for isochronous spins. In contrast, \( \tilde{R}_{SI} \neq \tilde{R}_{IS} \) in Eq. (F.8), because, as seen from Eqs. (E.22) – (E.24), \( R_{PP}^{\alpha} \) is not symmetric. The individual relaxation rates appearing in Eqs. (F.9), (F.11) and (F.12) are given in explicit form in Appendix E. All these rates connect single-spin modes and, according to the selection rule (E.15), they therefore only involve dipolar self-correlations. However, the ILRR is affected by cross-spin rates, such as \( R_{zz}^{IS} \), as well as cross-mode rates, such as \( R_{PP}^{z} \).

In the absence of spin \( P \), the \( ISP-IS \) results derived here must reduce to the previously obtained results for the symmetric two-spin case \( IS-IS \). With \( \omega_{D,IP} = \omega_{D,SP} = 0 \), the “cross-relaxation” supermatrix in Eq. (F.10) vanishes so Eq. (F.7) reduces to \( \tilde{R} = \langle R^\alpha \rangle \). For isochronous spins \( \langle R_{zz}^{II} \rangle = \langle R_{zz}^{SS} \rangle \) and \( \langle R_{zz}^{IS} \rangle = \langle R_{zz}^{SI} \rangle \), so Eqs. (F.7) – (F.9) yield the expected result

\[
\hat{R}_{dil}^{1,IS} = P_A \left[ \langle R_{zz}^{II} \rangle + \langle R_{zz}^{IS} \rangle \right] = \frac{2}{15} P_A \omega_{D,IS} \left[ j(\omega_0) + 4 j(2 \omega_0) \right],
\]

where we have also used Eq. (E.20).

If spin \( S \) is removed, the \( ISP-IS \) case degenerates into the asymmetric two-spin case \( IP-I \). With \( \omega_{D,IS} = \omega_{D,SP} = 0 \), \( R_{zz}^{IS} = R_{zz}^{SI} = R_{zz}^{SS} = 0 \) in Eq. (F.9) and \( R_{SP}^{\alpha} = 0 \) in Eq. (F.11). The only nonzero element of \( G^A(0) \) is then

\[
g_{11} = 1 - \frac{\langle R_{zz}^{II} \rangle}{\tau_A} + \frac{\langle R_{IP}^{\alpha} (R_{PP}^{\alpha} + i \omega_0 Q)^{-1} R_{IP}^{\alpha} \rangle}{\tau_A}.
\]

Replacing \( P \) by \( S \) to obtain the corresponding result for the previously treated \( IS-I \) case and noting that, for this case, \( \hat{R}_{dil}^{1} = (P_A/\tau_A) \left( 1 - g_{11} \right) \), we recover the expected result

\[
\hat{R}_{dil}^{1,I} = P_A \left[ \langle R_{zz}^{II} \rangle - \langle R_{IS}^{\alpha} (R_{SS}^{\alpha} + i \omega_0 Q)^{-1} R_{IS}^{\alpha} \rangle \right].
\]

The secular approximation is valid in the frequency range \( \omega_0 \gg \omega_0^2 \tau_A \), where the rapidly oscillating complex exponentials in Eq. (E.4) effectively eliminate all terms except those with \( M' = -M \). The selection rule (E.19) then shows that all single-spin cross-mode rates vanish.

As a result, Eqs. (F.7) – (F.12) yield

\[
\hat{R}_{dil}^{1,IS} = P_A \frac{2 (r_{II} r_{SS} - r_{IS}^2)}{(r_{II} + r_{SS} - 2 r_{IS})},
\]

where we have also used Eq. (E.20).
where

\[
\begin{align*}
    r_{II} &= \langle R_{zz}^{II} \rangle - \left( \frac{\langle R_{zz}^{IP} \rangle^2}{R_{zz}^{PP}} \right), \\
    r_{IS} &= \langle R_{zz}^{IS} \rangle - \left( \frac{\langle R_{zz}^{IP} R_{zz}^{SP} \rangle}{R_{zz}^{PP}} \right), \\
    r_{SS} &= \langle R_{zz}^{SS} \rangle - \left( \frac{\langle R_{zz}^{SP} \rangle^2}{R_{zz}^{PP}} \right).
\end{align*}
\]  

(2.17)

2. Exchange case $ISP - I$

If the basis is ordered with the 36 odd-parity operators before the 27 even operators, the supermatrices $L_A$ and $R_\alpha$ are block-diagonal whereas $\Delta_\alpha$ is anti-block-diagonal, as shown in Eqs. (E.12) and (E.13). Noting also that $K$ is diagonal, we can partition $\Lambda_\alpha$ in Eq. (F.2) as

\[
\begin{align*}
    \Lambda_\alpha &= \begin{pmatrix} (K_{aa} + R_{aa}^\alpha \tau_A + i L_{Z,aa}^A \tau_A) & i \Delta_{aa}^\alpha \tau_A \\
    i \Delta_{aa}^\alpha \tau_A & (K_{ss} + R_{ss}^\alpha \tau_A + i L_{Z,ss}^A \tau_A) \end{pmatrix},
\end{align*}
\]  

(2.18)

where the odd (or antisymmetric = a) and even (or symmetric = s) SIC parity is indicated by subscripts. The single-spin-$I$ subspace is contained within the odd-parity subspace so we only need the ‘aa’ block of the inverse $(\Lambda^\alpha)^{-1}$, which is

\[
(\Lambda^\alpha)_{aa}^{-1} = \left[ K_{aa} + (R_{aa}^\alpha + X_{aa}^\alpha + i L_{Z,aa}^A) \tau_A \right]^{-1},
\]  

(2.19)

with

\[
X_{aa}^\alpha = \Delta_{aa}^\alpha (K_{ss} + R_{ss}^\alpha \tau_A + i L_{Z,ss}^A \tau_A)^{-1} \Delta_{aa}^\alpha \tau_A.
\]  

(2.20)

We further partition the submatrices according to whether $K_{nn} = 1$ or 0. Ordering the basis so that, within the odd and even subspaces, basis operators with $K_{nn} = 1$ precede those with $K_{nn} = 0$, we have

\[
\begin{align*}
    K_{aa} &= \begin{bmatrix} 1_{aa11} & 0_{aa10} \\
    0_{aa01} & 0_{aa00} \end{bmatrix} \quad \text{and} \quad K_{ss} = \begin{bmatrix} 1_{ss11} & 0_{ss10} \\
    0_{ss01} & 0_{ss00} \end{bmatrix},
\end{align*}
\]  

(2.21)

where the subscripts indicate the subspace. Since there are six single-spin $S$ and $P$ operators in the ‘aa00’ subspace and nine two-spin-$SP$ operators in the ‘ss00’ subspace, it follows that $0_{aa00}$ is a $6 \times 6$ null matrix, while $0_{ss00}$ is a $9 \times 9$ null matrix. Furthermore, $1_{aa11}$ is a $30 \times 30$
identity matrix, while $1_{aa11}$ is a 18 × 18 identity matrix. We can thus partition the matrix to
be inverted in Eq. (F.19) as

$$
(\Lambda^\alpha)^{-1}_{aa} = \begin{bmatrix}
1_{aa11} + (R^\alpha_{aa11} + X^\alpha_{aa11} + i L^A_{\Delta,aa11}) \tau_A & (R^\alpha_{aa10} + X^\alpha_{aa10}) \tau_A \\
(R^\alpha_{aa01} + X^\alpha_{aa01}) \tau_A & (R^\alpha_{aa00} + X^\alpha_{aa00} + i L^A_{\Delta,aa00}) \tau_A
\end{bmatrix}^{-1}.
$$

We only need the $aa11$ block of $(\Lambda^\alpha)^{-1}_{aa}$, which is

$$
(\Lambda^\alpha)^{-1}_{aa11} = \begin{bmatrix}
1_{aa11} + (R^\alpha_{aa11} + X^\alpha_{aa11} + i L^A_{\Delta,aa11}) \tau_A \\
- (R^\alpha_{aa10} + X^\alpha_{aa10}) (R^\alpha_{aa00} + X^\alpha_{aa00} + i L^A_{\Delta,aa00})^{-1} (R^\alpha_{aa01} + X^\alpha_{aa01}) \tau_A
\end{bmatrix}^{-1}.
$$

Expanding to second order in $\omega_D \tau_A$ (the MN approximation) and performing the isotropic orientational average, we obtain with Eq. (F.1)

$$
G^A_{aa11}(0) = Z^{-1} - Z^{-1} [\langle R^\alpha_{aa11} \rangle + \langle X^\alpha_{aa11} \rangle] \tau_A Z^{-1}
$$

$$
+ Z^{-1} \langle (R^\alpha_{aa10} + X^\alpha_{aa10}) (R^\alpha_{aa00} + X^\alpha_{aa00} + i L^A_{\Delta,aa00})^{-1} (R^\alpha_{aa01} + X^\alpha_{aa01}) \rangle \tau_A Z^{-1},
$$

with $Z = 1_{aa11} + i L^A_{\Delta,aa11} \tau_A$. The ILRR is determined by the single element $g_{11} = \langle 1|G^A_{aa11}(0)|1 \rangle$ of this 30 × 30 supermatrix, where $|1 \rangle = B_1 = I_z/\sqrt{2}$. For this element, two simplifications can be made in Eq. (F.24). First, because $(1|L^A_{\Delta,aa11}|n) = (n|L^A_{\Delta,aa11}|1) = 0$, it follows that $(1|Z^{-1}|n) = (n|Z^{-1}|1) = \delta_{n1}$. Second, because $I_z$ commutes with $T^i_{SP}(SP)$, it follows from Eq. (E.10) that $(1|\Delta^\alpha_{aa}|n) = (n|\Delta^\alpha_{aa}|1) = 0$ and, in view of Eq. (F.20), that $(1|X^\alpha_{aa11}|1) = (1|X^\alpha_{aa10}|n) = (n|X^\alpha_{aa01}|1) = 0$. After these simplifications, Eq. (F.24) yields

$$
g_{11} = 1 - \langle \langle R^{II}_{zz} \rangle - \langle \Gamma^{II}_{zz} \rangle \rangle \tau_A,
$$

which is combined with Eq. (24) to yield

$$
\widehat{R}^{\text{dil}}_{1,I} = P_A \left[ \langle \langle R^{II}_{zz} \rangle - \langle \Gamma^{II}_{zz} \rangle \rangle \right].
$$

Here, we have defined the “cross relaxation” rate

$$
\Gamma^{II}_{zz} \equiv \langle 1|R^\alpha_{aa10} (R^\alpha_{aa00} + X^\alpha_{aa00} + i L^A_{\Delta,aa00})^{-1} R^\alpha_{aa01}|1 \rangle
$$

$$
= R^\alpha_{IS,IP} \begin{bmatrix}
(R^\alpha_{SS} + X^\alpha_{SS} + i \omega_0 Q) & X^\alpha_{SP} \\
X^\alpha_{PS} & (R^\alpha_{PP} + X^\alpha_{PP} + i \omega_0 Q)
\end{bmatrix}^{-1} R^\alpha_{IS,IP}^\dagger,
$$

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where \( Q = \text{diag}(0, 1, -1) \) and we have noted that \( R_{SP}^{\alpha} = 0 \) since the \( SP \) dipole coupling is static. Further, \( R_{IS,IP}^{\alpha} \equiv [R_{IS}^{\alpha,\alpha} \ R_{IP}^{\alpha,\alpha}] \) and \( R_{IS}^{\alpha,\alpha}, R_{IP}^{\alpha,\alpha}, R_{SS}^{\alpha,\alpha} \) and \( R_{PP}^{\alpha,\alpha} \) are defined in analogy with Eqs. (F.11) and (F.12). The individual relaxation rates appearing in these relaxation matrices are given in explicit form in Eqs. (E.21) – (E.25). All these rates connect single-spin modes and therefore, according to the selection rule Eq. (E.15), they only involve self-correlations.

The static \( SP \) dipole coupling affects the ILRR via the \( 3 \times 3 \) blocks \( X_{SS}^{\alpha}, X_{SP}^{\alpha}, X_{PS}^{\alpha} \) of the \( 6 \times 6 \) supermatrix \( X_{\alpha}^{\alpha} \) in the single-spin \( S \) and \( P \) subspace. According to Eq. (F.20), we have, for example,

\[
X_{SP}^{\alpha} = \Delta_{s,S}^{\alpha} (H_{ss}^{\alpha})^{-1} \Delta_{s,P}^{\alpha}, \tag{F.28}
\]

with \( H_{ss}^{\alpha} \equiv K_{ss}/\tau_A + R_{ss}^{\alpha} + iL_{ss}^{A} \). Note that subscript ‘s’ refers to the symmetric subspace spanned by the 27 two-spin operators in Table S1, while subscript \( S \) refers to the single-spin-S subspace, spanned by \( S_{z}/\sqrt{2}, -S_{z}/2 \) and \( S_{z}/2 \). It follows from Eqs. (E.8) and (E.10) that the supermatrices \( \Delta_{s,S}^{\alpha} \) and \( \Delta_{s,P}^{\alpha} \) have nonzero elements only within the ‘s0’ subspace spanned by the nine two-spin-\( SP \) operators. Consequently,

\[
X_{SP}^{\alpha} = \Delta_{s,s0}^{\alpha} (H_{ss0}^{\alpha})^{-1} \Delta_{s0,P}^{\alpha}. \tag{F.29}
\]

To obtain the quantity \((H_{ss}^{\alpha})_{00}^{-1}\), we partition \( H_{ss}^{\alpha} \) in the same way as for \( K_{ss} \) in Eq. (F.21),

\[
H_{ss}^{\alpha} = \begin{bmatrix}
1_{ss11}/\tau_A + R_{ss11}^{\alpha} + iL_{ss11}^{A} & R_{ss10}^{\alpha} \\
R_{ss01}^{\alpha} & (R_{ss00}^{\alpha} + iL_{ss00}^{A})
\end{bmatrix}. \tag{F.30}
\]

The required block of the inverse \((H_{ss}^{\alpha})_{00}^{-1}\) is

\[
(H_{ss}^{\alpha})_{00}^{-1} = \begin{bmatrix}
R_{ss00}^{\alpha} + iL_{ss00}^{A} - R_{ss01}^{\alpha} (1_{ss11}/\tau_A + R_{ss11}^{\alpha} + iL_{ss11}^{A})^{-1} R_{ss10}^{\alpha}
\end{bmatrix}^{-1}
= \begin{bmatrix}
(R_{ss00}^{\alpha} + iL_{ss00}^{A})^{-1}
\end{bmatrix}, \tag{F.31}
\]

where, in the last step, we have expanded to second order in \( \omega_D \tau_A \) to be consistent with the MN approximation. Finally, combination of Eqs. (F.29) and (F.31) yields

\[
\begin{bmatrix}
X_{SS}^{\alpha} & X_{SP}^{\alpha} \\
X_{PS}^{\alpha} & X_{PP}^{\alpha}
\end{bmatrix} = \Delta_{S/P,s0}^{\alpha} (R_{ss00}^{\alpha} + i\omega_0Q_{ss00})^{-1} \Delta_{S/P,s0}^{\alpha+}, \tag{F.32}
\]

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where $Q_{ss00} = \text{diag}(0,0,1,1,-1,-1,2,-2)$, $\Delta_{S/P,s0}^\alpha$ is the $6 \times 9$ matrix

$$
\Delta_{S/P,s0}^\alpha \equiv \begin{bmatrix}
\Delta_{S,s0}^\alpha \\
\Delta_{P,s0}^\alpha
\end{bmatrix},
$$

and $\Delta_{S/P,s0}^\alpha$ is its $9 \times 6$ Hermitian conjugate. This follows because the supermatrix $\Delta^\alpha$ is Hermitian, so $\Delta^\alpha_{s0,S} = \Delta^\alpha_{s0,S} \dagger$ and $\Delta^\alpha_{s0,P} = \Delta^\alpha_{s0,P} \dagger$. The static dipolar Liouvillian submatrices $\Delta_{S,s0}^\alpha$ and $\Delta_{P,s0}^\alpha$ can be obtained from Eq. (E.26). The elements of the $9 \times 9$ relaxation matrix $R_{ss00}^\alpha$, obtained from Eqs. (E.9) and (E.11), involve $IS$ and $IP$ self-correlations as well as $IS–IP$ distinct correlations.

We now consider several special cases. If we neglect the static $SP$ dipole coupling by setting $\omega_{D,SP} = 0$, then $\Delta^\alpha_{ss} = \Delta^\alpha_{ss} = 0$ so the antisymmetric and symmetric blocks in Eq. (F.18) are decoupled and the relaxation behavior is fully described by the single-spin modes. Because now $X^\alpha_{aa00} = 0$, Eq. (F.27) yields

$$
\Gamma_{zz}^{II} = R_{IS}^\alpha (R_{SS}^\alpha + i\omega_0 Q)^{-1} R_{IS}^\alpha \dagger + R_{IP}^\alpha (R_{PP}^\alpha + i\omega_0 Q)^{-1} R_{IP}^\alpha \dagger,
$$

showing, with Eq. (F.26), that the $IS$ and $IP$ dipole couplings contribute additively to the ILRR $\hat{R}_{1,I}^{\text{dil}}$. If both dipole couplings to spin $P$ are set to zero, then the $ISP–I$ case must reduce to the asymmetric two-spin case $IS–I$. With $\omega_{D,IP} = 0$, the second term in Eq. (F.34) vanishes and what remains is, as in Eq. (F.15), the result obtained previously for the $IS–I$ case, when specialized to isochronous spins.$^9$

If we (artificially) neglect cross-spin relaxation, so that $R_{IS}^\alpha = R_{IP}^\alpha = 0$, then Eqs. (F.26) and (F.27) show that

$$
\hat{R}_{1,I}^{\text{dil}} = P_A \langle R_{zz}^{II} \rangle = \frac{2}{45} P_A (\omega_{D,IS}^2 + \omega_{D,IP}^2) [j(0) + 3j(\omega_0) + 6j(2\omega_0)],
$$

where Eq. (E.20a) was also used. This (unphysical) result shows that cross-spin relaxation is necessary for the ILRR to approach zero at high field, as it must.

The secular approximation is valid in the frequency range $\omega_0 \gg \omega_0^2 / \tau_A$, where the rapidly oscillating complex exponentials in Eq. (E.4) effectively eliminate all terms except those with $M' = -M$. The selection rule (E.19) then shows that all single-spin cross-mode rates vanish. As a result, $R_{IS}^\alpha = [R_{zz}^{IIS}, 0, 0]$ and similarly for $R_{IP}^\alpha$ and Eq. (F.27) shows that
\( \Gamma_{zz}^{II} \) is a sum of four terms, each being the product of two such longitudinal cross-spin rates and the \( zz \) component of the corresponding block inverse. Moreover, these inverses are trivial because all the 3 \( \times \) 3 blocks in Eq. (F.27) are diagonal in the secular approximation. For \( R_{SS}^{\alpha} \) and \( R_{PP}^{\alpha} \), this follows from the vanishing of the cross-mode rates. For the four single-spin blocks of \( X_{aa00}^{\alpha} \), the diagonality follows from Eq. (F.32), where now \( R_{aa00}^{\alpha} \) is block-diagonal in \( Q \) (because \( M' = -M \) in Eq. (E.4)) and \( \Delta_{S,s0}^{\alpha} \) has only five nonzero elements, proportional to \( D_0 \) in Eq. (E.26). The latter simplification follows because the secular condition \( \omega_0 \gg \omega_D \) on the static \( SP \) coupling picks out the \( M = 0 \) term in Eq. (E.8). As a result of these simplifications in Eqs. (F.27) and (F.32), the “cross relaxation” rate in the secular approximation becomes

\[
\Gamma_{zz}^{II} = \frac{R_{zz}^{SS}(R_{zz}^{IP})^2 + R_{zz}^{PP}(R_{zz}^{IS})^2 + X(R_{zz}^{IS} + R_{zz}^{IP})^2}{R_{zz}^{SS} R_{zz}^{PP} + X(R_{zz}^{SS} + R_{zz}^{PP})}. \tag{F.36}
\]

Here, we have defined

\[
X \equiv \frac{2}{9} \left[ \omega_{D,SP} D_{00}^2 (\Omega_{SP}) \right]^2 \rho, \tag{F.37}
\]

where \( \rho \) is the ‘33’ element (corresponding to basis operator \( B_{19} \) in Table S1A) of the inverse of the 3 \( \times \) 3 \( Q = 0 \) block of \( R_{aa00}^{\alpha} \).
APPENDIX G: CHEMICAL SHIFTS

Here we generalize the BWR treatment of Sec. III A – C and Appendix F by allowing the Larmor frequencies of the three homonuclear spins to differ, as in the Zeeman Hamiltonian of Eq. (1). In the BWR theory, the ILRR depends on the chemical shifts in two ways.

First, there is an implicit dependence on the chemical shifts via the local relaxation supermatrix $R^\alpha$ in Eq. (26). This implicit shift effect essentially amounts to a replacement of $\omega_0$ in the spectral densities by a linear combination of the unequal Larmor frequencies. The effect is therefore present in the frequency range ($\omega_0 \approx 1/\tau_A$) of the primary dispersion, but not in the extreme-narrowing regime ($\omega_0 \tau_A \ll 1$). Chemical shifts displace the primary dispersion step along the frequency axis, but this displacement is of the same order as the chemical shifts themselves, so it is entirely negligible for homonuclear spins (with $\delta \ll 1$).

In the generalized BWR treatment, we can therefore retain the isochronous $R^\alpha$ derived in Appendix E.

Second, there is an explicit dependence on the chemical shifts via $L^A_Z$ and $L^B_Z$ in Eqs. (26) and (27), respectively. This explicit shift effect can be incorporated in the generalized BWR theory by using the non-isochronous Zeeman Liouvillians $L^A_Z$ and $L^B_Z$ derived in Appendix B. For the asymmetric exchange cases $ISP-IS$ and $ISP-I$, we find that the explicit shift effect amounts to displacements of the secondary dispersion steps of the same negligible order of magnitude as for the implicit shift effect on the primary dispersion step. In contrast, for the symmetric exchange case $ISP-IS$, the explicit shift effect is of higher order, giving rise to a novel inverted secondary dispersion step.

In summary, for homonuclear spins within the MN regime, chemical shifts have a significant effect on the longitudinal relaxation dispersion profile only for the symmetric exchange case $ISP-IS$. Under these conditions, the dispersion profiles computed from the generalized BWR theory developed here agree with the corresponding profiles obtained from the SLE theory, which rigorously incorporates all chemical shift effects. In Sec. IV D, we use the SLE theory to examine the effect of chemical shifts outside the MN regime.
1. Exchange case ISP–ISP

Combination of Eqs. (27) and (29), which are valid in the presence of chemical shifts, yields

\[ \tilde{U}^{BB}(0) = \frac{\tau_A}{P_A} [Z_B - Z_A^{-1} + Z_A^{-1} R^\alpha A Z_A^{-1} ]^{-1} , \]  

where

\[ Z_A \equiv 1 + iL^A_Z \tau_A , \]  

and similarly for \( Z_B \). All matrices appearing in Eqs. (G.1) and (G.2) are block-diagonal with respect to the projection index \( Q \) and with respect to odd/even SIC parity. It is therefore sufficient to consider the \( 10 \times 10 \) odd-parity block of these matrices. For isochronous spins, \( L^A_Z = L^B_Z = 0 \), so Eq. (G.1) reduces directly to Eq. (30).

According to Appendix B, the Zeeman Liouvillian in the presence of chemical shifts is

\[ L^A_Z = \omega^A_I \begin{pmatrix} 0 & 0 \\ 0 & D_A \end{pmatrix} , \]  

where the upper left null matrix 0 is \( 3 \times 3 \) and \( D_A \) is the \( 7 \times 7 \) matrix

\[ D_A = \begin{pmatrix} 0 & \delta_A' \\ \delta_A & 0 \end{pmatrix} , \]  

and where \( \delta_A' \) is the transpose of

\[ \delta_A = \begin{bmatrix} \frac{\sqrt{7}}{3} \delta_A S & \frac{1}{\sqrt{6}} (\delta_A S - 2 \delta_A P) & -\frac{\sqrt{7}}{6} \delta_A S \\ -\frac{2}{3} \delta_A S & \frac{\sqrt{5}}{6} (\delta_A S - 2 \delta_A P) & \frac{\sqrt{5}}{30} \delta_A S \\ 0 & -\frac{1}{2} \delta_A S & \frac{\sqrt{15}}{10} (\delta_A S - 2 \delta_A P) \end{bmatrix} . \]  

The supermatrix \( L^B_Z \) has a completely analogous form. In view of Eqs. (G.2) and (G.3),

\[ Z_B = \begin{pmatrix} 1 & 0 \\ 0 & Y_B \end{pmatrix} \quad \text{and} \quad Z_A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & Y_A^{-1} \end{pmatrix} , \]  

where

\[ Y_A \equiv 1 + i \omega^A_I \tau_A D_A , \]  

and similarly for \( Y_B \).

The orientationally averaged relaxation matrix \( < R^\alpha > \) in Eq. (G.1) may be partitioned into a \( 3 \times 3 \) longitudinal (L) block \( R_L \), a \( 7 \times 7 \) zero-quantum coherence (ZQC) block \( R_{ZQC} \),

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and two rectangular cross-mode blocks $R_{L/ZQC}$ and $R_{ZQC/L}$,
\[
\langle R^a \rangle = \begin{bmatrix}
R_L & R_{L/ZQC} \\
R_{ZQC/L} & R_{ZQC}
\end{bmatrix}.
\] (G.8)

Combination of Eqs. (G.1), (G.6), and (G.8) yields
\[
\tilde{U}^{BB}(0) = \frac{1}{P_A} \begin{bmatrix}
R_L & R_{L/ZQC} \cdot Y_A^{-1} \\
Y_A^{-1} R_{ZQC/L} & Y_A^{-1} R_{ZQC} \cdot Y_A^{-1} + (Y_B - Y_A^{-1})/\tau_A
\end{bmatrix}^{-1}.
\] (G.9)

We only need the $3 \times 3$ longitudinal block of $\tilde{U}^{BB}(0)$, obtained from Eq. (G.9) as
\[
\tilde{U}^{BB}_L(0) = [P_A R_L - P_A^2 R_{L/ZQC} (P_A R_{ZQC} + i W)^{-1} R_{ZQC/L}]^{-1},
\] (G.10)
where we have defined the frequency shift matrix
\[
W \equiv -i \frac{P_A}{\tau_A} Y_A (Y_B Y_A - 1).
\] (G.11)

So far, no approximations have been introduced. To obtain a more transparent expression for $W$, we note that $\epsilon \equiv \delta \omega_0 \tau_A \ll 1$ for homonuclear spins ($\delta \ll 1$) in the frequency range ($\omega_0 \tau_A \lesssim 10$) where longitudinal relaxation is effective. Here, $\delta$ is the largest of the shifts $\delta_S$ and $\delta_P$, and $\omega_0 \equiv \omega_I^A$, which differs from $\omega_I^B$ by a negligible amount of order $\delta$. To leading order in $\epsilon$, we obtain from Eqs. (G.7) and (G.11),
\[
W = \omega_0 (P_A D_A + P_B D_B).
\] (G.12)

To the same level approximation we can neglect the shift effect on the relaxation supermatrix $\langle R^a \rangle$ (see above), which can then be taken from Appendix E. We can now calculate $\hat{R}_{1,ISP}^{dil}$ by combing Eqs. (23c), (G.10), and (G.12), with $D_A$ and $D_B$ given by Eqs. (G.4) and (G.5).

In the absence of chemical shifts, $W = 0$ so Eq. (G.10) reduces (after division by $P_A$) to Eq. (33). As discussed in Sec. III A, the second term in Eq. (G.10), or Eq. (33), represents the effect of cross-mode relaxation driven by distinct correlations, which reduces $\hat{R}_{1,ISP}^{dil}$ below the value $\hat{R}_{1,ISP}^{dil, self}$ that would prevail if only self-correlations were included. It then follows from Eq. (G.10) that chemical shifts suppress the effect of cross-mode relaxation (and distinct correlations), thereby increasing $\hat{R}_{1,ISP}^{dil}$. As shown in Sec. IV D, this suppression can be nearly complete, but it only sets in above a certain frequency (indeed, $W$ vanishes
in the ZF limit). By considering the order or magnitude of the terms in Eqs. (G.10) and (G.12), this nonsecular decoupling (NSD) frequency can be identified as

\[ \omega_{\text{NSD}} = \frac{P_A \omega_D^2 \tau_A}{P_A \delta^A + P_B \delta^B}. \]  

We can now summarize the effect of chemical shifts on the dispersion profile \( \hat{R}_{1,\text{ISP}}^{\text{dil}}(\omega_0) \). At low frequencies, such that \( \omega_0 \ll \omega_{\text{NSD}} \), there is no effect. At \( \omega_0 \approx \omega_{\text{NSD}} \), there is an inverted secondary dispersion step as the (negative) contribution from distinct correlations in the L/ZQC cross-mode rates is partly lost. At higher frequencies, such that \( \omega_0 \gg \omega_{\text{NSD}} \), \( \hat{R}_{1,\text{ISP}}^{\text{dil}} \) remains larger than for isochronous spins but it never exceeds the ILRR \( \hat{R}_{1,\text{ISP}}^{\text{dil, self}} \) produced by self-correlations alone.

2. Exchange case \( \text{ISP} - \text{IS} \)

Because \( L^B_Z = 0 \) in the longitudinal two-spin subspace spanned by \( I_z \) and \( S_z \), the arguments leading from Eq. (27) to Eq. (34) remain valid for non-isochronous spins. We therefore only need to calculate the corresponding \( 2 \times 2 \) matrix \( G^A_L(0) \) in this longitudinal (L) subspace. Because also \( L^A_{Z,11} = 0 \) in this subspace, it follows that the development in Appendix F.1 remains valid, the only explicit effect of chemical shifts being the replacement of \( \omega_0 \) with \( \omega_0(1 + \delta^P) \) in Eq. (38). For isochronous spins, this explicit frequency dependence gives rise to an inverted secondary dispersion step at \( \omega_0 \approx \omega_0^2 \delta^A \tau_A \), above which cross-mode relaxation no longer contributes to \( \hat{R}_{1,\text{IS}}^{\text{dil}} \). The effect of the chemical shift (of the nonlabile spin \( P \); the shift of spin \( S \) has no effect here) is thus to displace the position on the frequency axis of this secondary dispersion step by a relative amount of order \( \delta^A \), which is negligible for homonuclear spins. In fact, this explicit shift effect is of the same order as the implicit shift effect on the primary dispersion step (at \( \omega_0 \approx 1/\tau_A \)) that we neglect by retaining the isochronous relaxation supermatrix \( R^\alpha \) from Appendix E. In conclusion, for exchange case \( \text{ISP} - \text{IS} \) the effects of chemical shifts are of higher order than for exchange case \( \text{ISP} - \text{ISP} \) and can be neglected altogether for homonuclear spins.
3. Exchange case ISP\textendash{}I

The development leading up to Eq. (41) remains valid in the presence of chemical shifts, because the $30 \times 30$ matrix $L_{Z,aa11}$ is block-diagonal with the first scalar block (corresponding to basis operator $I_z$) equal to zero. The two explicit occurrences of the Larmor frequency in Eqs. (42) and (43) are associated with the two secondary dispersion steps related to cross-mode relaxation and to the static $S$\textendash{}$P$ dipole coupling, respectively. The explicit shift effect in Eq. (42) amounts to replacement of $\omega_0$ by $\omega_0(1+\delta^A_S)$ in the $SS$ block and by $\omega_0(1+\delta^A_P)$ in the $PP$ block. The explicit shift effect in Eq. (43) amounts to replacement of the diagonal $9 \times 9$ matrix $Q_{ss00}$ by the block-diagonal matrix

$$Q_{ss00} = \begin{bmatrix}
Q_0 & 0 & 0 & 0 & 0 \\
0 & Q_1 & 0 & 0 & 0 \\
0 & 0 & Q_{-1} & 0 & 0 \\
0 & 0 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & Q_{-2}
\end{bmatrix}. \quad (G.14)$$

The diagonal blocks are

$$Q_0 = \frac{1}{\sqrt{3}} \begin{bmatrix}
0 & 0 & \sqrt{2} \delta_- \\
0 & 0 & \delta_- \\
\sqrt{2} \delta_- & \delta_- & 0
\end{bmatrix}, \quad (G.15a)$$

$$Q_{\pm1} = \frac{1}{2} \begin{bmatrix}
\pm(2+\delta_+) & \delta_- \\
\delta_- & \pm(2+\delta_+)
\end{bmatrix}, \quad (G.15b)$$

$$Q_{\pm2} = \pm(2+\delta_+), \quad (G.15c)$$

where $\delta_\pm \equiv \delta^A_S \pm \delta^A_P$.

The effect of chemical shifts is thus to displace the positions of the two secondary dispersion steps by relative amounts of order $\delta^A$, which is of the same order as the neglected implicit shift effect on the primary dispersion step. In conclusion, for exchange case ISP\textendash{}I, the effects of chemical shifts are of higher order than for exchange case ISP\textendash{}ISP and can be neglected altogether for homonuclear spins.
**APPENDIX H: ISP–I CASE IN THE MN REGIME:**  
**EFFECTS OF THE STATIC DIPOLE COUPLING**

Here we examine in detail how the static dipole coupling $\omega_{D,SP}$ affects the MRD profile for the ISP–I case in the dilute MN regime. According to Eq. (41), the ILRR of the labile $I$-spin is

$$\hat{R}_{1,I}^{\text{dil}}(\omega_0) = P_A \left[ \langle R_{zz}^{II}(\omega_0) \rangle - \langle \Gamma_{zz}^{II}(\omega_0) \rangle \right].$$  \hfill (H.1)

with the isotropically averaged longitudinal auto-spin relaxation rate $\langle R_{zz}^{II}(\omega_0) \rangle$ as in Eq. (E.20a) and the “cross relaxation” rate $\Gamma_{zz}^{II}(\omega_0)$ as in Eq. (42), which we now express as

$$\Gamma_{zz}^{II}(\omega_0) = \mathbf{\sigma} \mathbf{V}^{-1} \mathbf{\sigma}^\dagger,$$  \hfill (H.2)

with the $1 \times 6$ cross-spin relaxation matrix $\mathbf{\sigma}$

$$\mathbf{\sigma} = \begin{bmatrix} \sigma^{IS} & \sigma^{IP} \end{bmatrix} = \begin{bmatrix} \sigma_{zz}^{IS} & \sigma_{z+}^{IS} & \sigma_{z-}^{IS} & \sigma_{zz}^{IP} & \sigma_{z+}^{IP} & \sigma_{z-}^{IP} \end{bmatrix}.$$

The elements of $\mathbf{\sigma}^{IS}$ and $\mathbf{\sigma}^{IP}$ are proportional to $\omega_{D,IS}^2 \tau_A$ and $\omega_{D,IP}^2 \tau_A$, respectively. The $6 \times 6$ matrix $\mathbf{V}$, the inverse of which appears in Eq. (H.2), is

$$\mathbf{V} = \mathbf{X} + \mathbf{\rho} + i\omega_0 \mathbf{Q}_{11},$$  \hfill (H.4)

where $\mathbf{Q}_{11} = \text{diag}(0, 1, -1, 0, 1, -1)$ and

$$\mathbf{\rho} = \begin{bmatrix} \rho^{IS} & 0 \\ 0 & \rho^{IP} \end{bmatrix}.$$

The elements of the $3 \times 3$ auto-$S$-spin and auto-$P$-spin relaxation matrices $\mathbf{\rho}^{IS}$ and $\mathbf{\rho}^{IP}$ are proportional to $\omega_{D,IS}^2 \tau_A$ and $\omega_{D,IP}^2 \tau_A$, respectively. The static dipole coupling $\omega_{D,SP}$ affects the ILRR exclusively via the $6 \times 6$ matrix $\mathbf{X}$, given by

$$\mathbf{X} = \Delta (\mathbf{R}_{2\text{spin}} + i\omega_0 \mathbf{Q}_2)^{-1} \Delta^\dagger,$$  \hfill (H.6)

where $\mathbf{Q}_2 = \text{diag}(0, 0, 0, 1, -1, -1, 2, -2)$. The elements of the $9 \times 9$ relaxation matrix $\mathbf{R}_{2\text{spin}}$ in the two-spin-$SP$ subspace involve $IS$ and $IP$ self-correlations as well as $IS$–$IP$ distinct correlations. The elements of the $6 \times 9$ static dipolar Liouvillian matrix $\Delta$ are proportional to $\omega_{D,SP}$. 

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Figure S1a shows a typical $\hat{R}_{\text{dil}1,I}^{\text{ISP}}(\omega_0)$ dispersion profile for the $ISP - I$ case in the MN regime and Fig. S1b shows the component parts $\langle R_{zz}^{\text{II}}(\omega_0) \rangle$ and $\langle \Gamma_{zz}^{\text{II}}(\omega_0) \rangle$ in units of $\omega_{D,I}^2 \tau_A$, where $\omega_{D,I}^2 \equiv \omega_{D,I}^2 + \omega_{D,IP}^2$. If the static dipole coupling is artificially removed by setting $\omega_{D,SP} = 0$, then $\hat{R}_{\text{dil}1,I}^{\text{ISP}}(\omega_0)$ is reduced at all frequencies. In other words, the effect of the static dipole coupling $\omega_{D,SP}$ is to speed up the longitudinal relaxation of the labile $I$ spin. When $\omega_{D,SP} = 0$, the two fluctuating dipole couplings contribute additively so that $\hat{R}_{\text{dil}1,I}^{\text{ISP}} = \hat{R}_{\text{dil}1,I}^{\text{ISP}}(IS - I) + \hat{R}_{\text{dil}1,I}^{\text{ISP}}(IP - I)$, where $\hat{R}_{\text{dil}1,I}^{\text{ISP}}(IS - I)$ is the ILRR for the two-spin $IS - I$ case.

The $\hat{R}_{\text{dil}1,I}^{\text{ISP}}(\omega_0)$ profile exhibits a primary dispersion step at $\omega_0 \approx 1/\tau_A$, due mainly to the SDFs $j(\omega_0)$ and $j(2\omega_0)$ in $\langle R_{zz}^{\text{II}}(\omega_0) \rangle$. However, as seen from Fig. S1b, $\langle \Gamma_{zz}^{\text{II}}(\omega_0) \rangle$ also contributes to the primary dispersion step. In particular, it ensures that $\hat{R}_{\text{dil}1,I}^{\text{ISP}}(\omega_0)$ approaches zero asymptotically. This happens because the contributions from the zero-frequency SDF to $\langle R_{zz}^{\text{II}}(\omega_0) \rangle$ and $\langle \Gamma_{zz}^{\text{II}}(\omega_0) \rangle$ cancel out, so that $\langle R_{zz}^{\text{II}}(\infty) \rangle = \langle \Gamma_{zz}^{\text{II}}(\infty) \rangle$.

In addition, $\langle \Gamma_{zz}^{\text{II}}(\omega_0) \rangle$ exhibits a secondary dispersion step at a lower frequency. In the absence of a static dipole coupling (or if $\omega_{D,SP}$ is “weak”), the secondary dispersion occurs at $\omega_0 \approx \omega_{D,I}^2 \tau_A$, which then defines the (upper) boundary of the zero-field (ZF) regime. In the ZF regime, $I$-spin relaxation is slowed down by longitudinal-transverse cross-mode relaxation, which is eliminated by nonsecular decoupling when $\omega_0 \gg \omega_{D,I}^2 \tau_A$. In the presence of a (“strong”) static dipole coupling, this description is no longer valid. The secondary dispersion step at $\omega_0 \approx \omega_{D,I}^2 \tau_A$ then disappears and the ZF regime extends all the way up to $\omega_0 \approx \omega_{D,SP}$, where a new secondary dispersion step appears. Cross-mode relaxation occurs throughout this extended ZF regime, but it is modified by the static dipole coupling. Although relaxation in the ZF regime is modified by the static dipole coupling, it does not depend on the strength of this coupling as long as $\omega_{D,SP}$ is “strong” (in a sense to be defined).

In this Appendix, we rationalize this rather intricate relaxation behavior.
Figure S1: (a) Dispersion profile of $\tilde{R}_{1,I}^\text{dil}(\omega_0)$ for the $ISP-I$ case in the MN regime with (upper red) and without (lower red) static dipole coupling, and in the secular approximation (blue dashed). (b) Dispersion profiles for $\langle R_{zz}^H(\omega_0) \rangle$ (black dash-dot) and $\langle \Gamma_{zz}^H(\omega_0) \rangle$, both normalized by $\omega_{D,I}^2 \tau_A$. $\langle \Gamma_{zz}^H(\omega_0) \rangle$ is shown with (lower red) and without (upper red) static dipole coupling and in the secular approximation (blue dashed). Parameter values: $P_A = 10^{-3}$, $\tau_A = 10^{-7}$ s, $\omega_{D,IS} = 1 \times 10^5$ rad s$^{-1}$, $\beta_I = 50^\circ$, and $\beta_S = 70^\circ$, yielding $\omega_{D,IP} = 0.7828 \times 10^5$ rad s$^{-1}$ and $\omega_{D,SP} = 1.4449 \times 10^5$ rad s$^{-1}$.
1. Weak static dipole coupling

We characterize the strength of the static dipole coupling by the dimensionless parameter

\[ \epsilon \equiv \frac{\omega_{D,SP}}{\omega_{D,I}^2 \tau_A}. \]  \hspace{1cm} (H.7)

The static dipole coupling \( \omega_{D,SP} \) is said to be weak if \( \epsilon \ll 1 \) and strong if \( \epsilon \gg 1 \). For the homonuclear \( \text{ISP}-I \) case, the value of \( \epsilon \) depends on the internuclear geometry (\( \beta_I \) and \( \beta_S \)) and on the MN parameter \( \omega_{D,IS} \tau_A \), which is \( \ll 1 \) in the MN regime. For an equilateral triangle, where all three couplings are equal, \( \epsilon = 1/(2 \omega_{D,IS} \tau_A) \), which is \( \gg 1 \) in the MN regime. If either of the nonlabile spins is remote from the other two spins, one of the fluctuating dipole couplings is much larger than the static one, so that \( \epsilon \ll 1 \). For example, if spin \( P \) is located far from spins \( I \) and \( S \), the apex angle \( \beta_P \) is very small and \( \epsilon = \beta_P^3/(\omega_{D,IS} \tau_A) \). But as long as all three dipole couplings are of comparable magnitude, \( \epsilon \gg 1 \) in the MN regime. In this subsection we set \( \omega_{D,SP} = 0 \), but the results are the same for a finite \( \omega_{D,SP} \) as long as it is weak (\( \epsilon \ll 1 \)). As seen from Fig. S2, the weak-coupling limit of \( \langle \Gamma_{zz}^H(0) \rangle \) applies for \( \epsilon \lesssim 0.1 \), whereas the strong-coupling limit is a good approximation already for \( \epsilon \gtrsim 2 \).

If we set \( \omega_{D,SP} = 0 \) in Eq. (H.6), then \( X = 0 \), so \( V \) in Eq. (H.4) is block-diagonal and Eqs. (H.2) – (H.5) yield

\[ \langle \Gamma_{zz}^H(\omega_0) \rangle_0 = \langle \sigma^{IS} (\rho^{IS} + i \omega_0 Q_1)^{-1} \sigma^{IS\dagger} \rangle + \langle \sigma^{IP} (\rho^{IP} + i \omega_0 Q_1)^{-1} \sigma^{IP\dagger} \rangle, \] \hspace{1cm} (H.8)

where \( Q_1 = \text{diag}(0, 1, -1) \) and the 0 subscript reminds us that \( \omega_{D,SP} = 0 \). As expected, this is the sum of the \( \langle \Gamma_{zz}^H(\omega_0) \rangle \) expressions for the two-spin \( IS-I \) and \( IP-I \) cases. To identify the frequency of the secondary dispersion step, we introduce dimensionless quantities (labeled by a tilde) by writing \( \sigma^{IS} = \omega_{D,IS} \tau_A \tilde{\sigma}^{IS} \) and similarly for the other rates. Thus

\[ \langle \Gamma_{zz}^H(\omega_0) \rangle_0 = \omega_{D,IS}^2 \tau_A \left\langle \tilde{\sigma}^{IS} \left[ \tilde{\rho}^{IS} + i \frac{\omega_0}{\omega_{D,IS}^2 \tau_A} Q_1 \right]^{-1} \tilde{\sigma}^{IS\dagger} \right\rangle + \omega_{D,IP}^2 \tau_A \left\langle \tilde{\sigma}^{IP} \left[ \tilde{\rho}^{IP} + i \frac{\omega_0}{\omega_{D,IP}^2 \tau_A} Q_1 \right]^{-1} \tilde{\sigma}^{IP\dagger} \right\rangle. \] \hspace{1cm} (H.9)

Well below the primary dispersion step, such that \( \omega_0 \ll 1/\tau_A \), the cross-spin and auto-spin rates in Eq. (H.9) are all frequency-independent, so any dependence of \( \langle \Gamma_{zz}^H(\omega_0) \rangle_0 \) on \( \omega_0 \) in this regime must come from the explicit frequency factor in front of \( Q_1 \).
Figure S2: Variation of normalized $\langle \Gamma_{zz}(0) \rangle$ with $\omega_{D,SP}$ while all the other parameters are fixed at the values of Fig. S1. Vertical dashed lines correspond to indicated $\epsilon$ values.

Because the nonzero elements of $\tilde{\rho}^{IS}$, $\tilde{\rho}^{IP}$ and $Q_1$ are of order 1, the secondary dispersion step should appear when the pre-factor in front of $Q_1$ is of order 1, that is, at frequency $\omega_0 \approx \omega_{D,I}^2 \tau_A$. In the ZF regime, where $\omega_0 \ll \omega_{D,I}^2 \tau_A$, longitudinal-transverse cross-mode relaxation increases $\langle \Gamma_{zz}(\omega_0) \rangle_0$ to the value

$$\langle \Gamma_{zz}(0) \rangle_0 = \omega_{D,I}^2 \tau_A \left( \sigma^{IS} (\tilde{\rho}^{IS})^{-1} \sigma^{IS} \right)^\dagger + \omega_{D,I}^2 \tau_A \left( \sigma^{IP} (\tilde{\rho}^{IP})^{-1} \sigma^{IP} \right)^\dagger = \frac{14}{45} \omega_{D,I}^2 \tau_A, \quad (H.10)$$

where the last result has been derived before.\(^9\) In the LF regime, where $\omega_0 \gg \omega_{D,I}^2 \tau_A$, nonsecular decoupling of cross-mode relaxation reduces $\langle \Gamma_{zz}(\omega_0) \rangle_0$ to

$$\langle \Gamma_{zz}(\omega_0) \rangle_0 = \gamma \omega_{D,I}^2 \tau_A, \quad (H.11)$$

where $\gamma = 4/9 - c = 0.17666\ldots$, as shown before.\(^9\) This kind of secondary dispersion occurs only in spin systems with at least one nonlabile spin without strong static dipole couplings.
The secular approximation, which neglects longitudinal-transverse cross-mode relaxation, is justified at all frequencies such that \( \omega_0 \gg \omega_{D,I}^2 \tau_A \), where Eq. (H.9) yields

\[
\langle \Gamma_{zz}^{II}(\omega_0) \rangle_0^{sec} = \omega_{D,I}^2 \tau_A \gamma(\omega_0 \tau_A),
\]

where

\[
\gamma(\omega_0 \tau_A) \equiv \left\langle \frac{(\tilde{\sigma}_{zz}^X)^2}{\tilde{\rho}_{zz}^2} \right\rangle.
\]

This result is valid at all frequencies above the secondary dispersion step, even in the primary dispersion step (\( \omega_0 \approx 1/\tau_A \)), where the intrinsic rates \( \tilde{\sigma}_{zz}^X \) and \( \tilde{\rho}_{zz}^X \) are frequency-dependent. Note that \( \gamma(\omega_0 \tau_A) \) is independent of \( X \). As shown above, \( \gamma(0) = 0.17666 \ldots \) and \( \gamma(\infty) = 2/45 = 0.04444 \ldots \) (cf. Fig. S1).

2. Strong static dipole coupling

Introducing dimensionless quantities through \( \rho = \omega_{D,I}^2 \tau_A \tilde{\rho} \) and \( X = [\omega_{D,SP}^{2}/(\omega_{D,I}^2 \tau_A)] \tilde{X} \), we can write Eq. (H.4) as

\[
V = \omega_{D,I}^2 \tau_A \left[ \epsilon^2 \tilde{X} + \tilde{\rho} + i \frac{\omega_0}{\omega_{D,I}^2 \tau_A} Q_{11} \right],
\]

with \( \epsilon \) defined by Eq. (H.7). Similarly, with \( \Delta = \omega_{D,SP} \tilde{\Delta} \), Eq. (H.6) yields

\[
\tilde{X} = \tilde{\Delta} \left( \tilde{R}_{2\text{spin}} + i \frac{\omega_0}{\omega_{D,I}^2 \tau_A} Q_{22} \right)^{-1} \tilde{\Delta}^\dagger.
\]

In this subsection, we assume that the static dipole coupling is strong in the sense that \( \epsilon \gg 1 \).

2.1. Zero-field regime

In the extreme-narrowing regime, \( \omega_0 \tau_A \ll 1 \), the relaxation matrices \( \rho \) and \( R_{2\text{spin}} \) are Hermitian. It follows, therefore, from Eqs. (H.14) and (H.15) that also the 6 \times 6 matrices \( \tilde{X} \) and \( V \) are Hermitian in the limit \( \omega_0 = 0 \). For \( \epsilon \gg 1 \), as assumed here, \( V \) is strongly dominated by the first term within square brackets in Eq. (H.14). All elements of \( V \) are therefore proportional to \( \omega_{D,SP}^2 \). Accordingly, one might expect \( V^{-1} \) and, by way of Eq. (H.2), \( \langle \Gamma_{zz}^{II}(0) \rangle \), to be inversely proportional to \( \omega_{D,SP}^2 \). However, numerically we find that \( V^{-1} \) is independent of
\( \omega_{D,SP} \) when \( \epsilon \gg 1 \). This counter-intuitive behavior occurs because the matrix \( \tilde{X} \) is singular, with one zero eigenvalue.

Let \( \Lambda \) be the diagonal eigenvalue matrix and \( U \) the unitary matrix \((U^{-1} = U^\dagger)\) whose columns are the corresponding right eigenvectors of the Hermitian matrix \( \tilde{X} \). Then

\[
\tilde{X} U = U \Lambda, \tag{H.16}
\]

or

\[
\tilde{X} = U \Lambda U^\dagger. \tag{H.17}
\]

Since \( \tilde{X} \) is Hermitian, its eigenvalues are real. Numerical calculations confirm this and also show that the eigenvalues are non-negative and non-degenerate. Because of the normalization of \( \tilde{X} \), the eigenvalues are independent of \( \omega_{D,SP} \) and depend on the fluctuating dipole couplings only through their ratio \( \omega_{D,IS}/\omega_{D,IP} \). Finally, the eigenvalues are independent of the site orientation \( \Omega^\alpha \), but they do depend on the relative orientation of the dipole vectors as parametrized by \( \beta_I \) and \( \beta_S \). For convenience, we order the eigenvalues in ascending order, so that \( \lambda_1 = 0 \). The other five eigenvalues are of order 1.

Combining Eq. (H.14) with \( \omega_0 = 0 \) and Eq. (H.17), we obtain

\[
V = \omega_{D,I}^2 \tau_A U M U^\dagger, \tag{H.18}
\]

and, since \( V^{-1} = V^\dagger \),

\[
V^{-1} = \frac{1}{\omega_{D,I}^2 \tau_A} U M^{-1} U^\dagger, \tag{H.19}
\]

where

\[
M = \epsilon^2 \Lambda + U^\dagger \tilde{\rho} U. \tag{H.20}
\]

We partition the \( 6 \times 6 \) matrix \( M \) as

\[
M = \begin{bmatrix}
M_{11} & M_{1c} \\
M_{c1} & M_{cc}
\end{bmatrix}, \tag{H.21}
\]

where subscript \( c \) refers to the five-dimensional subspace spanned by eigenvectors \( u_2 - u_6 \). All nonzero elements of \( M \) are of order 1, except the diagonal elements of \( M_{cc} \), which are of order \( \epsilon^2 \gg 1 \). Consequently, all elements of \( M_{cc}^{-1} \) are \( \ll 1 \) and it follows from the standard expressions for the partitioned matrix inverse that all elements of \( M^{-1} \) are \( \ll 1 \), except \((M^{-1})_{11} = 1/M_{11} \), which is of order 1. We thus obtain from Eq. (H.19),

\[
V^{-1} = \frac{u_1 u_1^\dagger}{\omega_{D,I}^2 \tau_A M_{11}}, \tag{H.22}
\]
where $u_1$ is the (column) eigenvector corresponding to the eigenvalue $\lambda_1 = 0$. According to Eq. (H.20),

$$M_{11} = (U^\dagger \tilde{\rho} U)_{11} = u_1^\dagger \tilde{\rho} u_1. \quad (H.23)$$

Combining Eqs. (H.2), (H.22) and (H.23), and noting that $\sigma u_1 u_1^\dagger \sigma^\dagger = \sigma u_1 (\sigma u_1)^\dagger = |\sigma u_1|^2$, we obtain

$$\langle \Gamma^{II}_{zz}(0) \rangle = \langle \frac{|\sigma u_1|^2}{u_1^\dagger \rho u_1} \rangle. \quad (H.24)$$

The components of the eigenvectors $u_k$ refer to basis operators $\{S_z, S^+, S^-, P_z, P^+, P^-\}$ (disregarding normalization constants). The eigenvector $u_1$ has the special form

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} u_0 \\ u_0 \end{bmatrix}, \quad (H.25)$$

where $u_0$ is a $3 \times 1$ column vector and the numerical factor has been introduced to obtain the convenient normalization $u_0^\dagger u_0 = 1$. (Since $U$ is unitary, it follows that $u_1^\dagger u_1 = 1$.) Making use of Eqs. (H.3), (H.11) and (H.25), we can now write Eq. (H.24) as

$$\langle \Gamma^{II}_{zz}(0) \rangle = \langle \frac{|(\sigma^{IS} + \sigma^{IP}) u_0|^2}{u_0^\dagger (\rho^{IS} + \rho^{IP}) u_0} \rangle. \quad (H.26)$$

In Eq. (H.26), the eigenvector $u_0$ has the effect of projecting the cross-spin and auto-spin relaxation matrices onto the stationary ($\lambda_1 = 0$) subspace: $\sigma^{IS} u_0 = \sigma^{IS}_{z0} u_{0,1} + \sigma^{IS}_{z2} u_{0,2} + \sigma^{IS}_{z3} u_{0,3}$ and $u_0^\dagger \rho^{IS} u_0 = \sum_n \sum_p u_{0,n}^* u_{0,p} \rho^{IS}_{np}$. (For $\omega_0 = 0$, $\rho^{IS}$ is Hermitian, ensuring that, for any orientation, $u_0^\dagger \rho^{IS} u_0$, and thus $\Gamma^{II}_{zz}(0)$, is real-valued.) The eigenvector $u_0$ is independent of $\omega_{D,SP}$, but it depends on the ratio $\omega_{D,IS}/\omega_{D,IP}$ and on the orientation $\Omega_\alpha^{SP}$ of the static dipole vector. Consequently, while a strong static dipole coupling causes $\langle \Gamma^{II}_{zz}(0) \rangle$ in Eq. (H.26) to differ from $\langle \Gamma^{II}_{zz}(0) \rangle_0$ in Eq. (H.10), the actual value of $\langle \Gamma^{II}_{zz}(0) \rangle$ does not depend on $\omega_{D,SP}$ as long as this coupling is strong ($\epsilon \gg 1$). As we shall see, this behavior, illustrated for $\omega_0 = 0$ in Fig. S2, is actually observed throughout the ZF regime, that is, up to the secondary dispersion.
2.2. Secondary dispersion

For $\omega_0 > 0$, the explicit frequency term in Eq. (H.15) makes $\tilde{X}$ non-Hermitian. This follows by noting that the inverse of a Hermitian matrix is also Hermitian, so $\tilde{X}$ can be Hermitian only if $\{\tilde{R}_{2\text{spin}} + i [\omega_0 / (\omega_{D,I}^2 \tau_A)] Q_2\}$ is Hermitian. But this cannot be true for $\omega_0 \neq 0$, because the diagonal elements of a Hermitian matrix must be real.

Because $\tilde{X}$ is no longer Hermitian, the eigenvector matrix $U$ in Eq. (H.16) is not unitary so Eq. (H.17) must be replaced by

$$\tilde{X} = U \Lambda U^{-1}. \quad (H.27)$$

In general, both the eigenvalues $\lambda_k$ (as before, numbered in ascending order) and the eigenvectors $u_k$ (the columns of $U$) depend on the Larmor frequency $\omega_0$. However, the singular eigenvalue $\lambda_1 = 0$ occurs for all frequencies $\omega_0$ and the associated (orientation-dependent) eigenvector $u_1$ is independent of $\omega_0$, that is, it is the same as for $\omega_0 = 0$.

For $\omega_0 = 0$, when $\tilde{X}$ is Hermitian, all eigenvectors are orthonormal, $u_k^\dagger u_l = \delta_{kl}$. For $\omega_0 > 0$, when $\tilde{X}$ is non-Hermitian, the eigenvectors are still normalized, $u_k^\dagger u_k = 1$, but they are not orthogonal. The columns of $U$ are still the eigenvectors $u_k$, but the rows of $U^{-1}$ are not equal to $u_k^\dagger$. However, the eigenvector $u_1$ corresponding to the eigenvalue $\lambda_1 = 0$ defines a one-dimensional unitary subspace, such that $u_k^\dagger u_1 = u_1^\dagger u_k = 0$ for all $k \neq 1$. With the chosen ordering, the first column of $U$ is $u_1$, while the first row of $U^{-1}$ (or $U^\dagger$) is $u_1^\dagger$. Moreover, the half-eigenvector $u_0$, defined by Eq. (H.25), has the property (for any $\omega_0$ and any orientation)

$$|u_{0,2}| = |u_{0,3}|. \quad (H.28)$$

For $\omega_0 > 0$, we have in place of Eqs. (H.19) and (H.20)

$$V^{-1} = \frac{1}{\omega_{D,I}^2 \tau_A} \ U \ M^{-1} U^{-1}, \quad (H.29)$$

with

$$M \equiv \epsilon^2 \Lambda + U^{-1} \tilde{\rho} U + i \frac{\omega_0}{\omega_{D,I}^2 \tau_A} \ U^{-1} Q_{11} U. \quad (H.30)$$

We partition this $M$ matrix as in Eq. (H.21), where now

$$M_{11} = (U^{-1} \tilde{\rho} U)_{11} + i \frac{\omega_0}{\omega_{D,I}^2 \tau_A} \ (U^{-1} Q_{11} U)_{11} = u_1^\dagger \tilde{\rho} u_1, \quad (H.31)$$
where we have noted that Eq. (H.28) implies that

\[
(U^{-1} Q_{11} U)_{11} = 2 u_0^\dagger \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} u_0 = 2 (|u_{0,2}|^2 - |u_{0,3}|^2) = 0. \quad (H.32)
\]

Comparison of Eqs. (H.23) and (H.31) shows that \( M_{11} \) is independent of \( \omega_0 \) throughout the extreme-narrowing regime, where \( \omega_0 \ll 1/\tau_A \) so that \( \rho(\omega_0) = \rho(0) \).

As long as the nonzero diagonal elements, \( \epsilon^2 \lambda_k \) for \( k \geq 1 \), in the first term of Eq. (H.30) are much larger than any elements in the other terms, the arguments presented below Eq. (H.21) hold and Eqs. (H.22), (H.24) and (H.26) remain valid also for \( \omega_0 > 0 \). To establish quantitatively the extent of the ZF regime in the presence of a static dipole coupling, we consider first the regime \( \omega_0 \leq \omega_{D,I}^2 \tau_A \). According to Eq. (H.15), \( \tilde{X} \) is then of order 1, as are the eigenvalues \( \lambda_k \) for \( k \geq 1 \). All matrix elements in the second and third terms of Eq. (H.30) are of order 1 or less, whereas \( \epsilon^2 \lambda_k \gg 1 \) for \( k \geq 1 \) if the static dipole coupling is strong. The ZF result in Eq. (H.26) thus remains valid. In other words, a strong static dipole coupling abolishes the secondary dispersion at \( \omega_0 \approx \omega_{D,I}^2 \tau_A \).

To find the upper limit of the ZF regime, we consider now the regime \( \omega_0 \gg \omega_{D,I}^2 \tau_A \). According to Eq. (H.15), \( \tilde{X} \) is then of order \( \omega_{D,I}^2 \tau_A/\omega_0 \), as are the eigenvalues \( \lambda_k \) for \( k \geq 1 \). The order of magnitude of \( \epsilon^2 \lambda_k \) for \( k \geq 1 \) is then \( \omega_{D,I}^2 \omega_{D,I} / [\omega_0 \omega_{D,I}^2 \tau_A] \). With increasing Larmor frequency \( \omega_0 \), the first term in Eq. (H.30) thus decreases, whereas the third term increases. The condition for dominance of the first term is

\[
\omega_0 \ll \omega_{D,SP}, \quad (H.33)
\]

which defines the ZF regime in the presence of a strong static dipole coupling.
Let us now summarize the frequency dependence of $\langle \Gamma_{zz}^{II}(\omega_0) \rangle$ below the primary dispersion at $\omega_0 \approx 1/\tau_A$, as depicted in Fig. S1. For a weak ($\epsilon \ll 1$) static dipole coupling (or none), nonsecular decoupling of single-spin cross-mode relaxation occurs at $\omega_0 \approx \omega^{2}_{D,I} \tau_A$, giving rise to a secondary dispersion near this frequency, where $\langle \Gamma_{zz}^{II}(\omega_0) \rangle$ drops to the secular-approximation value $\langle \Gamma_{zz}^{II}(\omega_0) \rangle_0^{sec}$. In the presence of a strong ($\epsilon \gg 1$) static dipole coupling, there is no dispersion step at $\omega_0 \approx \omega^{2}_{D,I} \tau_A$ and cross-mode relaxation remains effective also at $\omega_0 > \omega^{2}_{D,I} \tau_A$, albeit modified by the static dipole coupling. Whereas for weak static dipole coupling the ZF regime extends up to $\omega^{2}_{D,I} \tau_A$, for strong static dipole coupling it extends all the way up to $\omega_{D,SP}$. Now there is a secondary dispersion at $\omega_0 \approx \omega_{D,SP}$, where $\langle \Gamma_{zz}^{II}(\omega_0) \rangle$ increases to the secular-approximation value $\langle \Gamma_{zz}^{II}(\omega_0) \rangle^{sec}$. This secondary dispersion is not caused by the SDFs, because we are still in the EN regime. Rather, it is caused by the explicit $Q_{11}$ term in Eq. (H.14), which decreases $\langle \Gamma_{zz}^{II}(\omega_0) \rangle$, and by the explicit $Q_2$ term in Eq. (H.15), which increases $\langle \Gamma_{zz}^{II}(\omega_0) \rangle$. As seen from Fig. S1, the latter effect dominates.

The secular approximation result in Eq. (46), which was derived under the assumption that $\omega_0 \gg \omega^{2}_{D,I} \tau_A$ and $\omega_0 \gg \omega_{D,SP}$, is valid without restriction on $\omega_{D,SP}$. For weak static dipole coupling ($\epsilon \ll 1$), we can neglect the $X$ terms in Eq. (46), which thereby reduces to Eq. (H.12). Conversely, for strong static dipole coupling ($\epsilon \gg 1$), the $X$ terms dominate and Eq. (46) reduces to

$$\langle \Gamma_{zz}^{II}(\omega_0) \rangle^{sec} = \left\langle \frac{(R_{zz}^{IS} + R_{zz}^{IP})^2}{(R_{zz}^{SS} + R_{zz}^{PP})} \right\rangle, \quad (H.34)$$

which is independent of $\omega_{D,SP}$. It can be shown that this result for $\langle \Gamma_{zz}^{II}(\omega_0) \rangle^{sec}$ is smaller than the result $\langle \Gamma_{zz}^{II}(\omega_0) \rangle_0^{sec}$ in Eq. (H.12), as is evident from Fig. S1.

The secular approximation simplifies the cross-spin relaxation matrix $\sigma$ in Eq. (H.3), the auto-spin relaxation matrix $\rho$ in Eq. (H.5), and the static dipole matrix $X$ in Eq. (H.6). By applying the secular approximation selectively to these matrices, we find that, for strong static dipole coupling, the secular approximation acts mainly via $X$, whereas, for weak static dipole coupling, the effect is on $\sigma$ and $\rho$ (since $X$ is not involved). The different nonsecular decoupling effects for weak and strong dipole coupling explains why the $\tilde{R}_{1,1}^{dil}(\omega_0)$ dispersion is inverted for weak but not for strong static dipole coupling (Fig. S1).
References


